14.125.303: Biomedical Transport Phenomena

LECTURE 6 (02.06.2020)

ADMINISTRATIVE:

- Canvas uploads from class: Lecture05, HWs
- Make Up HWs (Sickness, Clicker malfunction) before Exam I

TODAY:

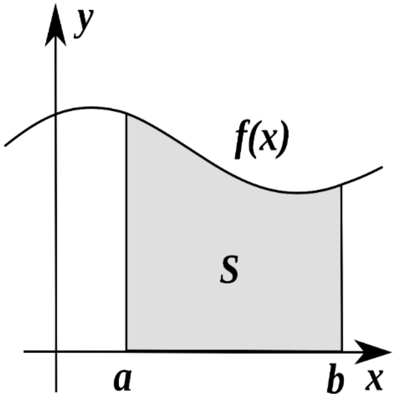
- Review HW4
- Mass Conservation
- Control Volumes
- Small Group Problems (3)

*****Q2 Next Lecture*****

LEC	DATE	GENERAL TOPICS PER LECTURE	ASSESSMENT	READING
1	01-21	Introduction & Overview		
2	01-23	Review of Physical Properties; Phases of Matter	HW1	Ch1
3	01-28	Review of Mathematics and Dimensions (SI)	HW2	Ch1
4	01-30	Forces, Energy Work; Ideal vs Real Models	HW3	Ch2
5	02-04	Fluid Static Forces, Properties, and Applications	Q1	Ch2,Ch3
6	02-06	Control Mass & Volume- Conservation Equations	HW4	Ch3,Ch4
7	02-11	Conservation of Mass	Q2	Ch4
8	02-13	Inviscid Flows	HW5	Ch4
9	02-18	Recitation-In-Class Review	Q3	
10	**02-20**	**EXAM I (States and Equilibrium)**		

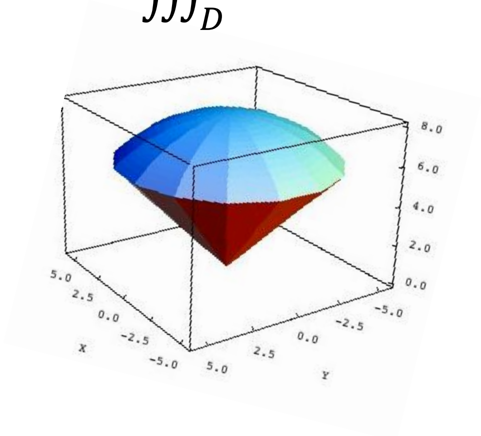
Integrals

$$F(x) = \int_b^a f(x)dx = \{F(b) - F(a)\}$$



$$F(x,y) = \int_b^a f(x,y)dxdy$$

$$F(x) = \iiint_D^0 f(x,y,z)dxdydz$$



We relate volume integrals and surface integrals using Gauss Theorem

$$\int_A \vec{v}\cdot\vec{n}dA = \int_V \vec{\nabla}\cdot\vec{v}dV$$

***Gauss Theorem**:*

- *Outward flux through a closed surface is equal to the volume integral of the divergence inside the surface*
- *The sum of all sources of the field in a region gives the net flux out of the region*

$$\underbrace{\frac{\partial}{\partial t}\int_{CV}\rho\, d\forall}_{\text{Time rate of increase of mass in CV}} + \underbrace{\int_{CS}\rho\vec{V}\cdot d\vec{A}}_{\text{Net influx of mass crossing CS}} = 0$$

$$\underbrace{\left(\frac{\partial M}{\partial t}\right)_{CV}}_{\text{Time rate of increase of mass in CV}} = \underbrace{\sum \dot{m}_{\text{IN}} - \sum \dot{m}_{\text{OUT}}}_{\text{Net influx of mass crossing CS}}$$

(ρ V A)

What does steady-state mean?

Time effects are negligible

All temporal derivatives go to ZERO

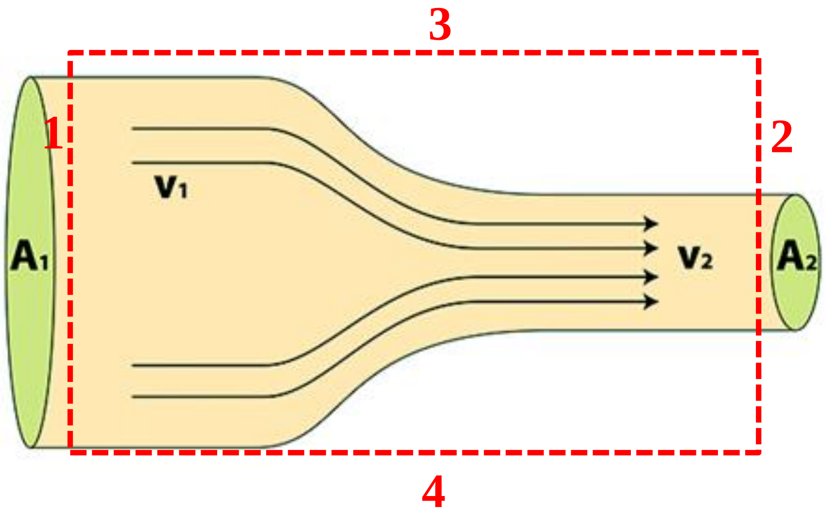
$$\frac{\partial}{\partial t} = 0 \qquad \nabla \cdot (\rho\, \underline{\mathbf{v}}) = 0$$

What does incompressible mean?

Density is constant

All spatial derivatives of ρ go to ZERO

$$\frac{\partial \rho}{\partial x, \partial y, \partial z} = 0 \qquad \nabla \cdot \underline{\mathbf{v}} = 0$$



1. Streamlines should follow the contour of the channel inner surfaces.

2. CV should have defined surface conditions.

Mass conservation *always* holds

$$\frac{\partial}{\partial t}\int_{CV}\rho\, d\mathcal{V} + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0$$

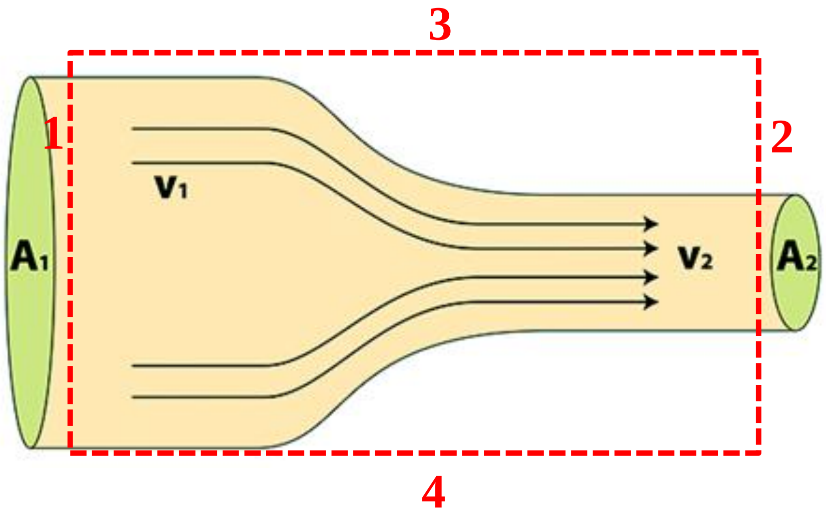
Assume state and incompressible

$$\int_{CS}\rho\vec{V}\cdot d\vec{A} = 0$$

Apply this along surfaces 1- 4

$$\rho_1 V_1 A_1 - \rho_2 V_2 A_2 = 0$$

$$\boxed{\frac{V_1 A_1}{A_2} = V_2}$$



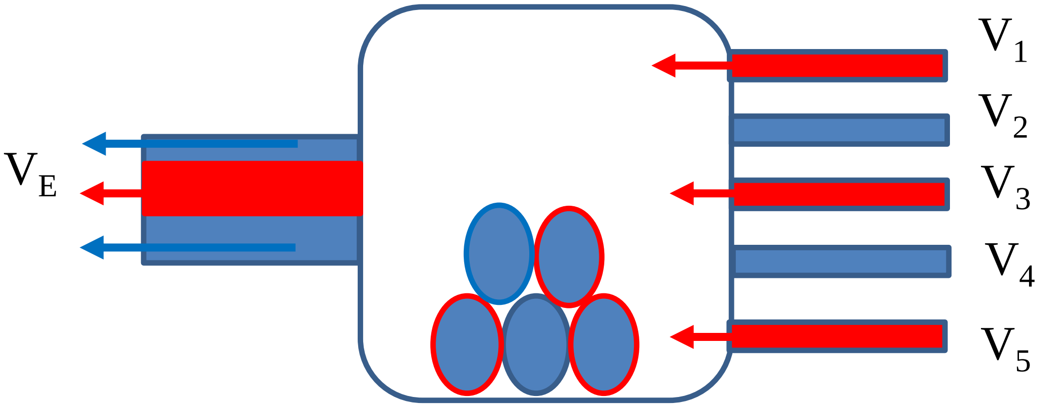
Can we apply Bernoulli?

$$z_2 + \frac{v_2^2}{2g} + \frac{P_2}{\rho g} = z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\rho g}$$

- Inviscid; Steady and Uniform flow; Incompressible; Zero heat transfer
- Need to know P_2! But after mass conservation we can find P_2.
- Tip: Use mass conservation to derive velocities

$$\boxed{\frac{V_1 A_1}{A_2} = V_2}$$

Spheres of collagen hydrogels are placed within a closed reservoir maintained at pressure $5P_0$. Five different gaseous flows of catalytic reactants are then pumped into the reservoir through tubing of radius, R, each. The flows exit the chamber through one larger tubing of radius 3R. You perform an analysis to determine the exit velocity, V_E. **Which of the following statements is TRUE?**



A. Bernoulli's principle cannot be applied because the fluids are viscous.

B. The solution requires knowledge of the inlet pressures for the tubings.

C. The inlet velocities must be equal because their inlet radii are equal.

D. The exit velocity V_E is a linear function of the inlet velocities, $V_1 – V_5$.

E. All of the Above

A. Bernoulli's principle cannot be applied because the fluids are viscous.

$$z_2 + \frac{v_2^2}{2g} + \frac{P_2}{\rho g} = z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\rho g}$$

- Inviscid; Steady and Uniform flow; Incompressible; Zero heat transfer
- Cannot be applied here because of catalytic gaseous flow (<u>compressible</u>! And <u>reactive</u>!)

B. The solution requires knowledge of the inlet pressures for the tubings.

$$\boxed{\frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho \, d\forall + \int_{\mathrm{CS}} \rho \vec{V} \cdot d\vec{A} = 0}$$

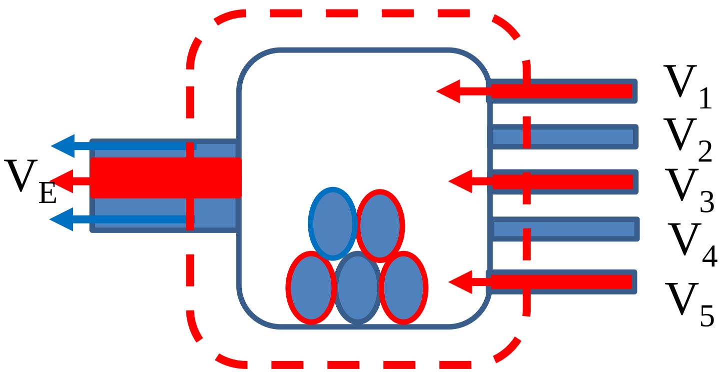
$$\boxed{\left(\frac{\partial M}{\partial t}\right)_{CV} = \sum \dot{m}_{\mathrm{IN}} - \sum \dot{m}_{\mathrm{OUT}}}$$

- <u>Mass conservation</u> does not require explicit knowledge of pressure

C. The inlet velocities must be equal because their inlet radii are equal.

- Volume flow rate, Q = V A, will be equal [m^3/s]
- Mass flow rate is defined as $\dot{m} = \rho V A$ [kg/s]
- Multiple catalytic flows may have different densities, ρ

D. The exit velocity V_E is a linear function of the inlet velocities, $V_1 – V_5$.



$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$$

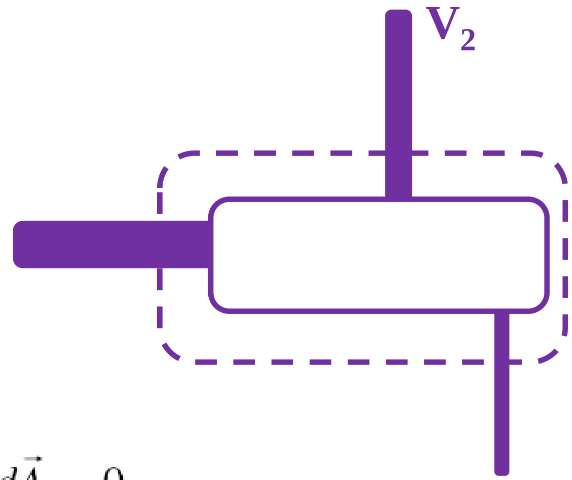
$$\boxed{\rho_1 V_1 A_1 + \rho_2 V_2 A_2 + \rho_3 V_3 A_3 + \rho_4 V_4 4 + \rho_5 V_5 A_5 - \rho_E V_E A_E = 0}$$

E. All of the above.

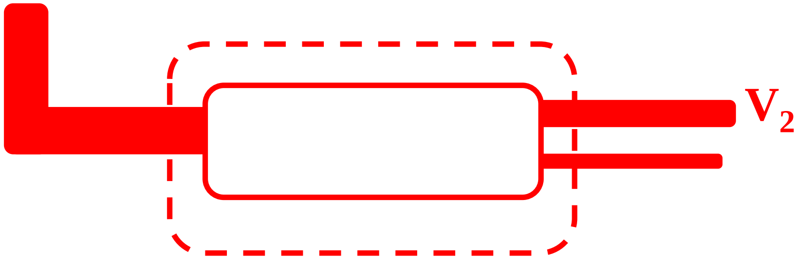
QUESTIONS??

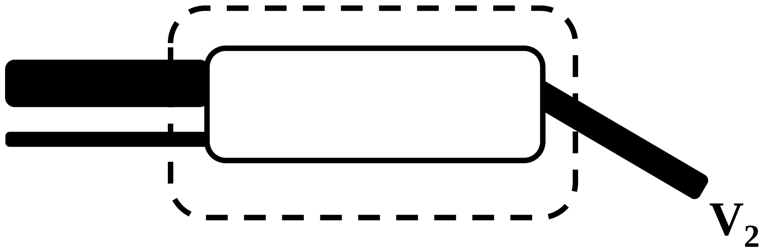
PROBLEM 1

Determine how the expression for exit velocity, V_2, is different in each of these configurations:



$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$$

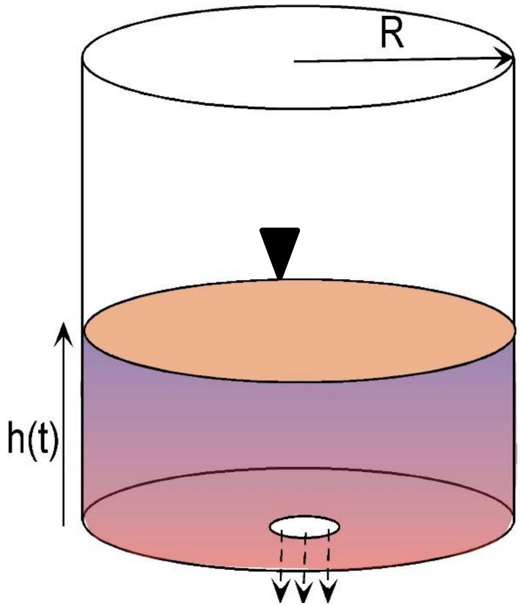




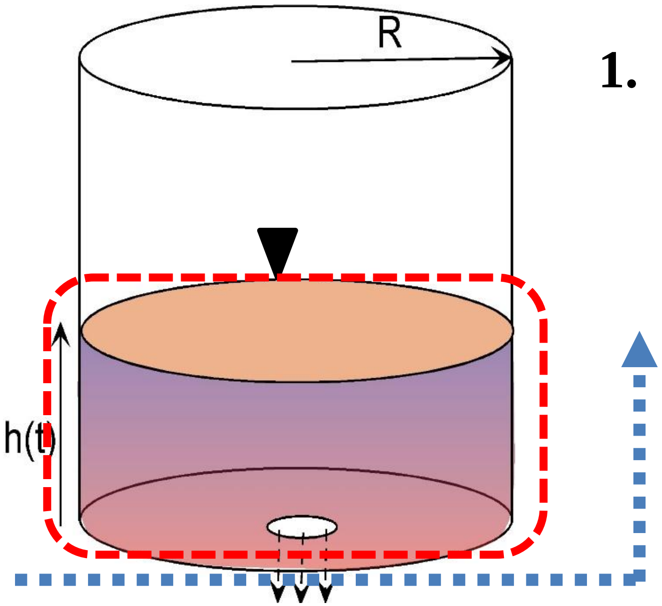
How is the pressure at the exit, P_2, altered by these configurations?

PROBLEM 2

A large circular reservoir of radius R is filled to a height h_o with a fluid of density ρ at time t=0. At initial times, a hole of small, fixed area A is created at the bottom. The bottom efflux of fluid is very, very slow.



1. Derive an expression for the exit velocity, V_E.
2. Calculate the time t_F necessary to empty the reservoir.



1. **Derive an expression for the exit velocity, V_E.**

$$\frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0$$

$$\frac{\partial}{\partial t}\int[\rho(\pi R^2\, dh)] + \int_{CS2}^{CS1}\rho V dA = 0$$

$$\frac{\partial}{\partial t}[\rho(\pi R^2\, h(t))] \quad + [\cancel{(\rho V A_{IN})} - (\rho V A_{EXIT})] = 0$$

V~0 in large reservoir

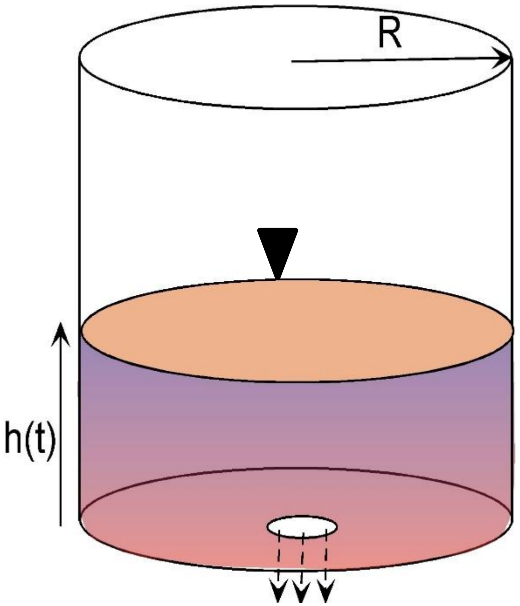
How to find efflux velocity, V_E?

<u>Assume</u>: Inviscid, incompressible, Newtonian fluid in adiabatic and <u>isobaric</u> flow

$$z_1 + \cancel{\frac{V_1^2}{2g}} + \cancel{\frac{P_1}{\rho g}} = \cancel{z_2} + \frac{V_2^2}{2g} + \cancel{\frac{P_2}{\rho g}}$$

$V_1 \sim 0$ $\quad$ $z_E = 0$ IRF $\quad$ $P_1 = P_2 = P_0$

$$\boxed{V_E = \sqrt{2g(h)}}$$



2. Calculate the time t_F necessary to empty the reservoir.

$$\frac{\partial}{\partial t}[\rho(\pi R^2\, h(t))] \quad + [0 - (\rho V A_E)] = 0$$

$$V_E = \sqrt{2g(h)}$$

$$[\rho(\pi R^2)]\frac{dh}{dt} \quad = \quad \left[(\rho A_E)(\sqrt{2gh}\right]$$

$$\int \frac{dh}{\sqrt{h}} \quad = \int \frac{(\rho A_E)(\sqrt{2g}\, dt}{\rho(\pi R^2)}$$

Boundary Condition:
h @ ($t = t_F$) = 0

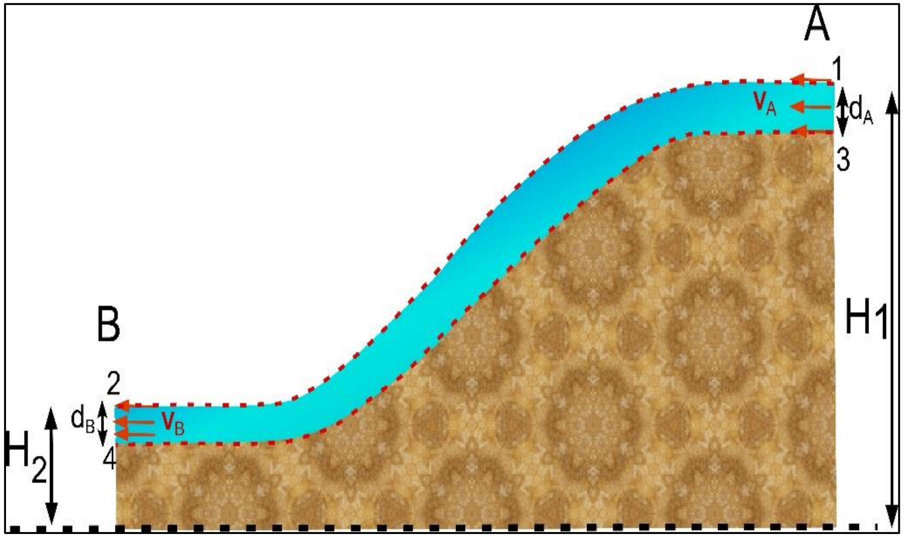
$$2\left(h_o^{1/2} - h^{\frac{1}{2}}(t\ )\right) = \frac{(\rho A_E)(\sqrt{2g}}{\rho(\pi R^2)}\{t\}$$

$$\boxed{\frac{(\pi R^2)\sqrt{2h_o}}{(A_E)(\sqrt{g})} = \{t_F\}}$$

QUESTIONS??

RECITATION PROBLEM

Consider the steady flow of a river of an inviscid, incompressible fluid with density ρ down a mountain. Two streamlines for the flow are shown in the diagram below. Points 1 and 3 are at cross section A at the top and points 2 and 4 are at cross section B at the bottom. The atmospheric pressure is P_o. At the top, the velocity is uniform across the cross section A as shown with a river velocity v_A and a river height d_A. At the bottom (cross section B) the velocity is also uniform with velocity v_B and height d_B. The reference plane for gravity is shown, and the heights H_1 and H_2 are above the reference plane. The river width (out of the page) is W.



1. Deduce which velocity is greater, v_A or v_B.

2. Derive a relationship for v_A/v_B in terms of d_B/d_A.

3. Calculate the pressure at points 3 and 4.

(i)

$$p_1+\frac{1}{2}\rho v_A^2+\rho gH_1=p_2+\frac{1}{2}\rho v_B^2+\rho gH_2$$

$$\left\{\begin{array}{l} p_1=p_o \text{ and } p_2=p_o \\ p_o+\frac{1}{2}\rho v_A^2+\rho gH_1=p_o+\frac{1}{2}\rho v_B^2+\rho gH_2 \\ \frac{1}{2}\rho v_B^2-\frac{1}{2}\rho v_A^2=\rho gH_1-\rho gH_2 \end{array}\right\}$$

$$\left\{\begin{array}{l} H_1>H_2 \\ \therefore v_B>v_A \text{ The river speeds up as it flows} \\ \text{down the hill, converting gravitational} \\ \text{to kinetic energy of fluid translation.} \end{array}\right\}$$

(ii)

$$\rho Wd_B v_B=\rho Wd_a v_A$$

$$\frac{v_B}{v_A}=\frac{d_A}{d_B}$$

$$\left\{\begin{array}{l} \frac{1}{2}\rho v_B^2-\frac{1}{2}\rho v_A^2=\rho gH_1-\rho gH_2 \\ 1-\frac{v_A^2}{v_B^2}=\frac{2g\{H_1-H_2\}}{v_B^2} \\ 1-\frac{d_B^2}{d_A^2}=\frac{2g\{H_1-H_2\}}{v_B^2} \end{array}\right\}$$

$$\left\{\begin{array}{l} \frac{1}{2}r v_B^2-\frac{1}{2}r v_A^2=r gH_1-r gH_2 \\ 1-\frac{v_A^2}{v_B^2}=\frac{2g\{H_1-H_2\}}{v_B^2} \\ 1-\frac{d_B^2}{d_A^2}=\frac{2g\{H_1-H_2\}}{v_B^2} \end{array}\right\} \quad (5)$$

$$\left\{\begin{array}{l} H_1>H_2 \\ \backslash \;\; d_B<d_A \text{ The river height} \\ \text{decreases as it flows down the} \\ \text{hill, since its speed increases} \\ \text{and mass is conserved} \end{array}\right\} \quad (5)$$

(iii)

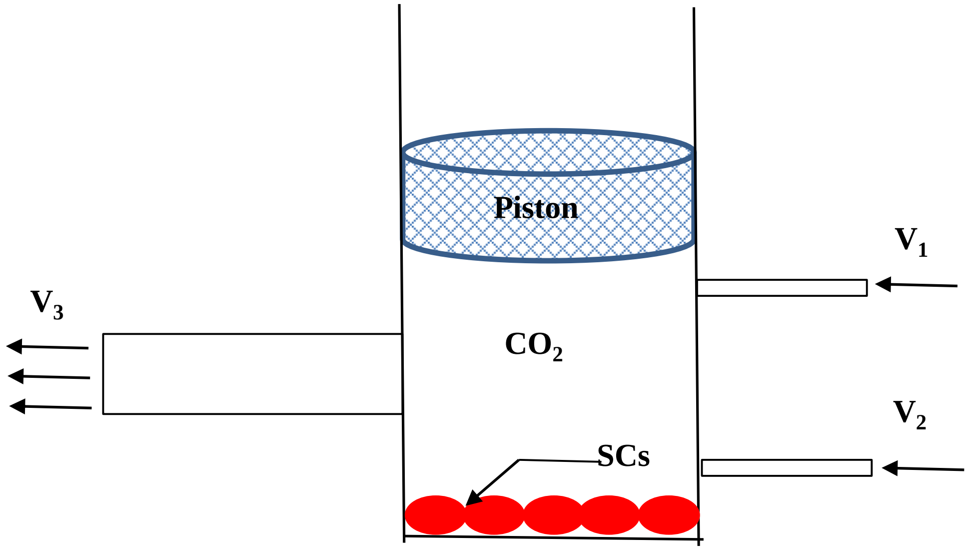
$$p_3+\frac{1}{2}r v_A^2+r g(H_1-d_A)=p_4+\frac{1}{2}r v_B^2+r g\left(H_2-d_B\right)$$

$$p_3=p_o+r gd_A \qquad p_4=p_o+r gd_B$$

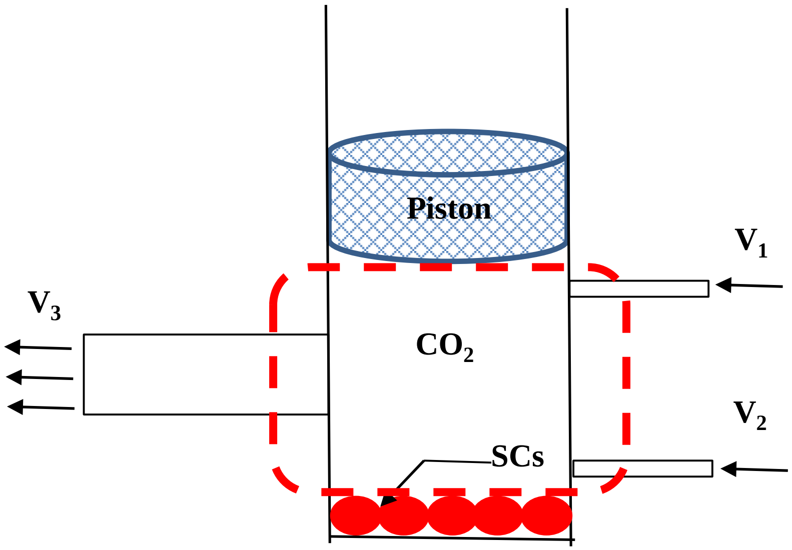
$$\left\{\begin{array}{l} p_o+r gd_A+\frac{1}{2}r v_A^2+r g(H_1-d_A)=p_o+r gd_B+\frac{1}{2}r v_B^2+r g\left(H_2-d_B\right) \\ p_o+r v_A^2+r g(H_1)=p_o+\frac{1}{2}r v_B^2+r g\left(H_2\right) \text{ same as (i)} \end{array}\right\}$$

QUIZ 1 from SP2019

Cells (SCs) are plated at the bottom of a reservoir of cross sectional area, A, open to the atmosphere. $C0_2$ enters the reservoir from tubing 1 (cross-sectional area A_1) and tubing 2 (cross-sectional area A_2) with velocity V_1 and V_2, respectively. The CO_2 exits the device through tubing 3 (of cross-sectional area A_3). A solid piston of mass, M_p, with cross sectional area A rests upon the open reservoir. The flow of $C0_2$ can be modeled as incompressible.

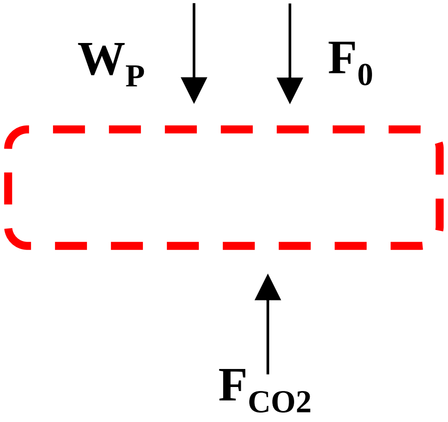


1. **Determine an expression for the average pressure of the $C0_2$ in the reservoir, P_{CO2}.**
2. **A colleague measures the inlet velocities V_1 and V_2 to be twice that of the exit velocity, each. Derive an expression for the outlet area, A_1.**



1. Determine an expression for the average pressure of the C0$_2$ in the reservoir, P_{CO2}.

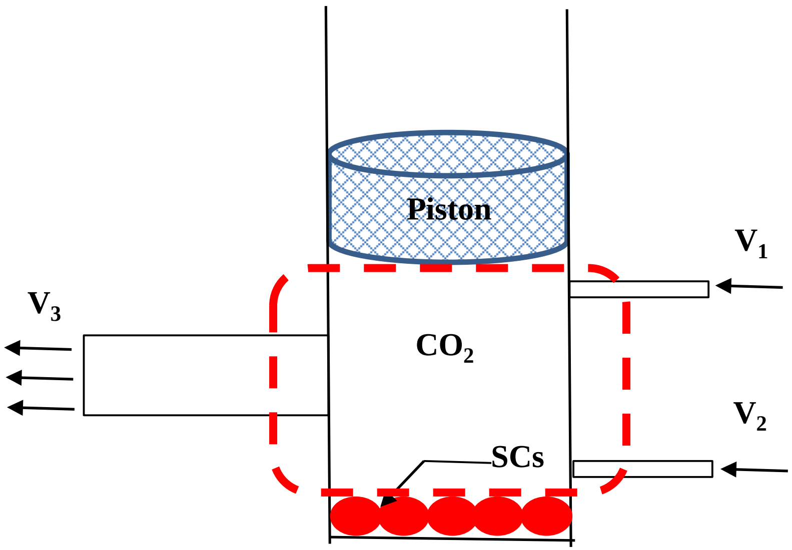
Assume *reservoir is of very large volume and inlet/outlet flows are steady and ideal (i.e. do not contribute to vertical pressure forces)*



$$\sum \boldsymbol{F} = m\,\boldsymbol{a}$$

$$\boxed{P_{CO2}\,A = M\,g + P_o\,A}$$

2. A colleague measures the inlet velocities V_1 and V_2 to be twice that of the exit velocity, each. Derive an expression for the outlet area, A_1.



Means: $V_1 = V_2 = 2\,V_3$

Also: Nothing is known about pressures of inlet or outlet flows.

$$\cancel{\frac{\partial}{\partial t}\int_{CV}\rho\, d\forall} + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0$$

Steady-State

$$\frac{\partial}{\partial t} = 0$$

$$\rho_1 V_1 A_1 + \rho_2 V_2 A_2 = \rho_3 V_3 A_3$$

$$\boxed{\rho_1(2V_3)A_1 + \rho_2(2V_3)A_2 = \rho_3 V_3 A_3}$$

QUESTIONS??

NEXT TIME:

- Q2 at 1:40PM
- Mass Conservation

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