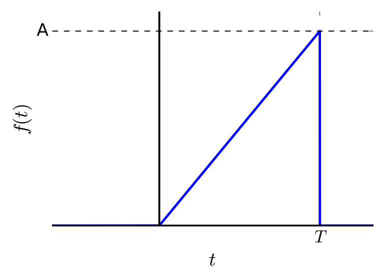
ECE 345: Linear Systems and Signals
Professors Sarwate and El Rouayheb

Fall 2020
Due: Monday, December 7, 2020

Solutions for Homework 7

Problem 1 (Computing CTFTs from the definition, SSTA 5.39). Find the CTFT of the following signal with $A = 5.0$ and $T = 3.0$:



Solution:

First we need a formula for that signal:

$$f(t) = \begin{cases} 0 & t \leq 0 \\ \frac{A}{T}t & 0 \leq t \leq T \\ 0 & t \geq T \end{cases} \tag{1}$$

So now we apply the CTFT definition and use integration by parts:

$$\begin{aligned} F(j\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt & (2) \\ &= \int_0^T \frac{A}{T}te^{-j\omega t} & (3) \\ &= \frac{A}{T}\left[\frac{jte^{-j\omega t}}{\omega}\right]_0^T - \frac{j}{\omega}\int_0^T e^{-j\omega t}dt & (4) \\ &= \frac{A}{T\omega^2}[e^{-j\omega T}(1 + j\omega T) - 1]. & (5) \end{aligned}$$

Now we can substitute in the values for A and T:

$$F(j\omega) = \frac{5}{3\omega^2}[(1 + j3\omega)e^{-j3\omega} - 1]. \tag{6}$$

Problem 2 (Using CTFT properties, OW 4.6). Suppose that $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(j\omega)$. Then find the CTFT of the following signals using CTFT properties.

(a) $x_1(t) = x(1 - t) + x(-1 - t)$

(b) $x_2(t) = x(3t - 6)$

(c) $x_3(t) = \frac{d^2}{dt^2}x(t - 1)$

Solution:

(a) First use time-reversal : $x(-t) \xleftrightarrow{\text{CTFT}} X(-j\omega)$. This gives

$$x(-t+1) \xleftrightarrow{\text{CTFT}} e^{-j\omega} X(-j\omega) \tag{7}$$

$$x(-t-1) \xleftrightarrow{\text{CTFT}} e^{j\omega} X(-j\omega) \tag{8}$$

So by linearity

$$x_1(t) \xleftrightarrow{\text{CTFT}} e^{-j\omega} X(-j\omega) + e^{j\omega} X(-j\omega) = 2X(-j\omega)\cos(\omega) \tag{9}$$

(b) The first thing to do here is scaling:

$$x(3t) \xleftrightarrow{\text{CTFT}} \frac{1}{3} X\left(j\frac{\omega}{3}\right) \tag{10}$$

Now sending $t \to t-2$ is time-shifting:

$$x_2(t) \xleftrightarrow{\text{CTFT}} e^{-2j\omega} \frac{1}{3} X\left(j\frac{\omega}{3}\right) \tag{11}$$

(c) For this one we need the differentiation property applied twice, followed by a delay by 1:

$$\frac{d}{dt} x(t) \xleftrightarrow{\text{CTFT}} j\omega X(j\omega) \tag{12}$$

$$\frac{d^2}{dt^2} x(t) \xleftrightarrow{\text{CTFT}} -\omega^2 X(j\omega) \tag{13}$$

$$x_3(t) \xleftrightarrow{\text{CTFT}} -\omega^2 X(j\omega) e^{-j\omega} \tag{14}$$

Problem 3 (Using CTFT properties II)**.** Suppose that $x(t) \xleftrightarrow{\text{CTFT}} X(j\omega)$. Then find the time domain signals corresponding to the following CTFTs.

(a) $X_1(j\omega) = \frac{4}{3} e^{-j2\omega/3} X(-j\omega/3)$

(b) $X_2(j\omega) = \frac{1}{3} e^{j2(\omega-2)} X^* \left(j\frac{\omega-2}{3}\right)$

Solution:

The trick is to figure out how the order of signal manipulations that lead from $X(j\omega)$ to $X_i(j\omega)$. You can do this going "forwards" (from $X(j\omega)$ to $X_i(j\omega)$) or "backwards" by undoing each operation.

(a) We can start with the following:

$$x(t) \xleftrightarrow{\text{CTFT}} X(j\omega) \tag{15}$$

$$x(-t) \xleftrightarrow{\text{CTFT}} X(-j\omega) \tag{16}$$

$$x(-t+2) \xleftrightarrow{\text{CTFT}} e^{-j2\omega} X(-j\omega) \tag{17}$$

$$x(-3t+2) \xleftrightarrow{\text{CTFT}} \frac{1}{3} e^{-j2\omega/3} X(-j\omega/3) \tag{18}$$

$$4x(-3t+2) \xleftrightarrow{\text{CTFT}} \frac{4}{3} e^{-j2\omega/3} X(-j\omega/3) \tag{19}$$

So $x_1(t) = 4x(-3t+2)$.

(b) Working forwards again:

$$x(t) \xleftrightarrow{\text{CTFT}} X(j\omega) \tag{20}$$
$$x^*(t) \xleftrightarrow{\text{CTFT}} X^*(-j\omega) \tag{21}$$
$$x^*(-t) \xleftrightarrow{\text{CTFT}} X^*(j\omega) \tag{22}$$
$$x^*(-3t) \xleftrightarrow{\text{CTFT}} \frac{1}{3}X^*(j\omega/3) \tag{23}$$
$$x^*(-3t) \xleftrightarrow{\text{CTFT}} \frac{1}{3}X^*(j\omega/3) \tag{24}$$
$$x^*(-3(t+2)) \xleftrightarrow{\text{CTFT}} \frac{1}{3}e^{j2\omega}X^*(j\omega/3) \tag{25}$$
$$e^{j2t}x^*(-3(t+2)) \xleftrightarrow{\text{CTFT}} \frac{1}{3}e^{j2(\omega-2)}X^*(j(\omega-2)/3) \tag{26}$$

Problem 4 (Computing CTFTs, OW 4.21)**.** Compute the CTFT for the following signals.

(a) $e^{-3|t|}\sin(2t)$

(b) $\frac{\sin(\pi t)}{(\pi t)} \cdot \frac{\sin(2\pi(t-1))}{\pi(t-1)}$

(c) $te^{-2t}\sin(4t)u(t)$

Solution:

(a) We can look up the following transforms from the table:

$$e^{-3|t|} \xleftrightarrow{\text{CTFT}} \frac{6}{9+\omega^2} \tag{27}$$
$$\sin(2t) \xleftrightarrow{\text{CTFT}} \frac{-\pi}{j}\delta(\omega+2) + \frac{\pi}{j}\delta(\omega-2) \tag{28}$$

Using the convolution property,

$$e^{-3|t|}\sin(2t) \xleftrightarrow{\text{CTFT}} \frac{1}{2\pi}\left(\frac{6}{9+\omega^2} * (\frac{-\pi}{j}\delta(\omega+2) + \frac{\pi}{j}\delta(\omega-2))\right) \tag{29}$$
$$= \frac{-3/j}{9+(\omega+2)^2} + \frac{3/j}{9+(\omega-2)^2} \tag{30}$$

(b) For the first and second terms,

$$\frac{\sin(\pi t)}{(\pi t)} \xleftrightarrow{\text{CTFT}} X_1(j\omega) = \begin{cases} 1 & |\omega| < \pi \\ 0 & \text{otherwise} \end{cases} \tag{31}$$
$$\frac{\sin(2\pi t)}{(\pi t)} \xleftrightarrow{\text{CTFT}} = \begin{cases} 1 & |\omega| < 2\pi \\ 0 & \text{otherwise} \end{cases} \tag{32}$$
$$\frac{\sin(2\pi(t-1))}{\pi(t-1)} \xleftrightarrow{\text{CTFT}} = X_2(j\omega) = \begin{cases} e^{-j\omega} & |\omega| < 2\pi \\ 0 & \text{otherwise} \end{cases} \tag{33}$$
$$\tag{34}$$

So now we convolve the two:

$$X(j\omega) = \frac{1}{2\pi}X_1(j\omega) * X_2(j\omega). \tag{35}$$

There's no real choice but to do this one by hand, but since $X_1(j\omega)$ is just a box from $-\pi$ to π, the convolution is just the integral of $X_2(j\omega)$ over intervals of length 2π.

Flipping $X_1(j\omega)$, we get for $-3\pi < \omega < -\pi$:

$$\begin{aligned}
X(j\omega) &= \frac{1}{2\pi}\int_{-2\pi}^{\omega+\pi} e^{-j\phi}d\phi && (36)\\
&= \frac{1}{2\pi}\left[\frac{-1}{j}e^{-j\phi}\right]_{-2\pi}^{\omega+\pi} && (37)\\
&= \frac{1}{2\pi}\left(\frac{-1}{j}e^{-j\omega}e^{-j\pi}+\frac{1}{j}\right) && (38)\\
&= \frac{-1}{2\pi}j(e^{-j\omega}+1). && (39)
\end{aligned}$$

For $-\pi < \omega < \pi$:

$$\begin{aligned}
X(j\omega) &= \frac{1}{2\pi}\int_{\omega-\pi}^{\omega+\pi} e^{-j\phi}d\phi && (40)\\
&= \frac{1}{2\pi}\left[\frac{-1}{j}e^{-j\phi}\right]_{\omega-\pi}^{\omega+\pi} && (41)\\
&= \frac{1}{2\pi}\left(\frac{-1}{j}e^{-j\omega}e^{-j\pi}+\frac{1}{j}e^{-j\omega}e^{j\pi}\right) && (42)\\
&= \frac{1}{2\pi}e^{-j\omega}(-j+j) && (43)\\
&= 0. && (44)
\end{aligned}$$

For $\pi < \omega < 3\pi$:

$$\begin{aligned}
X(j\omega) &= \frac{1}{2\pi}\int_{\omega-\pi}^{2\pi} e^{-j\phi}d\phi && (45)\\
&= \frac{1}{2\pi}\left[\frac{-1}{j}e^{-j\phi}\right]_{\omega-\pi}^{2\pi} && (46)\\
&= \frac{1}{2\pi}\left(\frac{-1}{j}+\frac{1}{j}e^{-j\omega}e^{j\pi}\right) && (47)\\
&= \frac{1}{2\pi}j(e^{-j\omega}+1). && (48)
\end{aligned}$$

And for $|\omega| > 3\pi$ the output is 0.

(c) Eulerizing:

$$x(t) = \frac{1}{2j}te^{-2t}e^{j4t}u(t) - \frac{1}{2j}te^{-2t}e^{-j4t}u(t), \qquad (49)$$

so using the table:

$$X(j\omega) = \frac{1/2j}{(2-j4+j\omega)^2} - \frac{1/2j}{(2+j4+j\omega)^2}. \qquad (50)$$

Problem 5 (Even and odd parts, OW 4.9)**.** Suppose

$$x(t) = \begin{cases} 0 & |t| > 1 \\ \frac{t+1}{2} & -1 \le t \le 1 \end{cases} \qquad (51)$$

First, sketch the function so that you know what you are looking at.

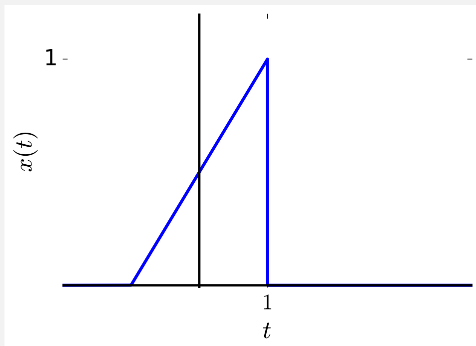
(a) Find the CTFT $X(j\omega)$

(b) Find $x_{\text{even}}(t) = \frac{x(t)+x(-t)}{2}$ and $x_{\text{odd}}(t) = \frac{x(t)-x(-t)}{2}$.

(c) Find the CTFT of the even and odd parts of $x(t)$ and check that they correspond to the real and imaginary parts of $X(j\omega)$

Solution:

Here's what the function looks like:



(a) There are lots of ways to solve this. If you solved an earlier problem involving a ramp (e.g. on this homework) then you could apply transforms to that by using scaling and delay. The solution here is chosen to highlight the use of the integration properties of the CTFT. The signal shown here can be expressed in terms of the integral of $y(t) = \frac{1}{2}\,\text{rect}(t/2)$:

$$y(t) = \begin{cases} 1/2 & |t| < 1 \\ 0 & \text{otherwise} \end{cases} \tag{52}$$

To see this, consider

$$\int_{-\infty}^{t} y(\tau) d\tau \tag{53}$$

This grows linearly as a function of t until $t = 1$. So then we need to subtract of $u(t-1)$:

$$x(t) = \int_{-\infty}^{t} y(\tau) d\tau - u(t-1). \tag{54}$$

We have

$$y(t) \xleftrightarrow{\text{CTFT}} \frac{\sin(\omega)}{\omega} \tag{55}$$

$$\int_{0}^{t} y(t)dt \xleftrightarrow{\text{CTFT}} \frac{Y(j\omega)}{j\omega} + \pi Y(j0)\delta(\omega) = \frac{\sin(\omega)}{j\omega^2} + \pi\delta(\omega) \tag{56}$$

$$u(t) \xleftrightarrow{\text{CTFT}} \pi\delta(\omega) + \frac{1}{j\omega} \tag{57}$$

$$u(t-1) \xleftrightarrow{\text{CTFT}} e^{-j\omega}\left(\pi\delta(\omega) + \frac{1}{j\omega}\right) = \pi\delta(\omega) + e^{-j\omega}\frac{1}{j\omega} \tag{58}$$

Here we are using CTFT properties and the fact that the $\delta(\omega)$ is zero everywhere except $\omega = 0$.

So

$$X(j\omega) = \frac{\sin(\omega)}{j\omega^2} + \pi\delta(\omega) - \pi\delta(\omega) + e^{-j\omega}\frac{1}{j\omega} \tag{59}$$

$$= \frac{\sin(\omega)}{j\omega^2} - \frac{e^{-j\omega}}{j\omega} \tag{60}$$

(b) We can find the even part:

$$x_{\text{even}}(t) = \frac{x(t) + x(-t)}{2} = y(t) \tag{61}$$

You can see this graphically or algebraically: $\frac{1}{2}\left(\frac{t+1}{2} + \frac{-t+1}{2}\right) = \frac{1}{2}$. For the odd part,

$$x_{\text{odd}}(t) = \frac{x(t) - x(-t)}{2} = \frac{1}{2}\left(\frac{t+1}{2} - \frac{-t+1}{2}\right) = \frac{t}{2} \tag{62}$$

for $|t| < 1$. You could also take $x(t) - x_{\text{even}}(t)$.

(c) Since $x(t)$ is real, the CTFT of the even part goes to the real part of $X(j\omega)$ and the CTFT of the odd part goes to the imaginary part of $X(j\omega)$:

$$x_{\text{even}}(t) \overset{\text{CTFT}}{\longleftrightarrow} Y(j\omega) = \frac{\sin(\omega)}{\omega} = \mathfrak{Re}\{X(j\omega)\} \tag{63}$$

$$x_{\text{odd}}(t) \overset{\text{CTFT}}{\longleftrightarrow} \frac{\sin(\omega)}{j\omega^2} - \frac{\cos(\omega)}{j\omega} = \mathfrak{Im}\{X(j\omega)\} \tag{64}$$

$$\tag{65}$$

We can see this by Eulerizing the $e^{-j\omega}$ and taking $X(j\omega) - X_{\text{even}}(j\omega)$.

Problem 6 (System identification is not so easy)**.** Consider the following three impulse responses:

$$h_1(t) = u(t) \tag{66}$$

$$h_2(t) = -2\delta(t) + 5e^{-2t}u(t) \tag{67}$$

$$h_3(t) = 2e^{-t}u(t) \tag{68}$$

Using the CTFT, do the following.

1. Find the output of these systems to the input $\cos(t)$.
2. Find the output of these systems to the input $\cos(2t)$.
3. Find another system whose output with input $\cos(t)$ is the same as $h_1(t)$.
4. Find another system whose output with input $\cos(2t)$ is the same as $h_2(t)$.

What this means is that if you have an LTI system with a wire going in and a wire going out, just looking at the output to a single cosine input will not let you figure out ("identify") the system impulse response.

Solution:

Using the CTFT,

$$H_1(j\omega) = \pi\delta(\omega) + \frac{1}{j\omega} \tag{69}$$

$$H_2(j\omega) = -2 + \frac{5}{2+j\omega} = -2 + \frac{10-5j\omega}{4+\omega^2} = \frac{2-5j\omega-2\omega^2}{4+\omega^2} \tag{70}$$

$$H_3(j\omega) = \frac{2}{1+j\omega} = \frac{2-j2\omega}{1+\omega^2} \tag{71}$$

We also have $\cos(at) = \frac{1}{2}e^{jat} + \frac{1}{2}e^{-jat}$. Since complex exponentials are eigenfunctions of LTI systems, we just need to evaluate the CTFT at $\pm a$:

1. Here $a = 1$ so for the first system,

$$H_1(j) = \pi\delta(1) + \frac{1}{j} = -j, \tag{72}$$

$$H_1(-j) = \pi\delta(-1) + \frac{1}{-j} = j, \tag{73}$$

$$y_1(t) = \frac{-j}{2}e^{j\omega t} + \frac{j}{2}e^{-j\omega t} = \sin(t) \tag{74}$$

For the second system

$$H_2(1) = \frac{-5j}{5} = -j \tag{75}$$

$$H_2(-1) = j \tag{76}$$

$$y_2(t) = \sin(t) \tag{77}$$

For the third system

$$H_3(1) = 1 - j \tag{78}$$

$$H_3(-1) = 1 + j \tag{79}$$

$$y_3(t) = \cos(t) + \sin(t) \tag{80}$$

2. Here $a = 2$ so for the first system,

$$H_1(2) = \pi\delta(2) + \frac{1}{j2} = \frac{-j}{2}, \tag{81}$$

$$H_1(-2) = \pi\delta(-2) + \frac{1}{-j2} = \frac{j}{2}, \tag{82}$$

$$y_1(t) = \frac{-j}{4}e^{j\omega t} + \frac{j}{4}e^{-j\omega t} = \frac{1}{2}\sin(t) \tag{83}$$

For the second system

$$H_2(2) = \frac{-6-10j}{8} = \frac{3}{4} - \frac{5}{4}j \tag{84}$$

$$H_2(-2) = \frac{3}{4} + \frac{5}{4}j \tag{85}$$

$$y_2(t) = \frac{3}{4}\cos(t) + \frac{5}{4}\sin(t) \tag{86}$$

For the third system

$$H_3(2) = \frac{2}{5} - \frac{4}{5}j \tag{87}$$
$$H_3(-2) = \frac{2}{5} + \frac{4}{5}j \tag{88}$$
$$y_3(t) = \frac{2}{5}\cos(t) + \frac{4}{5}\sin(t) \tag{89}$$

3. Any system whose value $H(-1) = j$ and $H(1) = -j$ will work.
4. Any system whose value $H(-2) = \frac{3}{4} + \frac{5}{4}j$ and $H(2) = \frac{3}{4} - \frac{5}{4}j$ will work.

Problem 7 (Filtering)**.** One thing that differs between textbooks is how they define the sinc function. For example, in the textbook (SSTA p.219), they define

$$\mathrm{rect}(t) = \begin{cases} 1 & \frac{-1}{2} \le t \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \mathrm{sinc}(t) = \frac{\sin(t)}{t} \tag{90}$$

and the CTFT pairs

$$\mathrm{rect}\left(\frac{t}{\tau}\right) \xleftrightarrow{\text{CTFT}} \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right) \tag{91}$$
$$\frac{W}{\pi}\,\mathrm{sinc}(Wt) \xleftrightarrow{\text{CTFT}} \mathrm{rect}\left(\frac{\omega}{2W}\right) \tag{92}$$

Unfortunately, MATLAB's `sinc`(t) function is defined to be $\frac{\sin(\pi t)}{\pi t}$, which leads to all sorts of confusion.

In this problem we will use the above defenitions of rect and sinc. Suppose that you apply the periodic signal

$$x(t) = 1 + 2\cos(5\pi t) + 3\sin(8\pi t) \tag{93}$$

to an LTI system with impulse response

$$h(t) = 8\,\mathrm{sinc}(4t)\cos(4\pi t). \tag{94}$$

(a) Sketch $x(t)$ and $h(t)$.

(b) Compute the fundamental radian frequency of $x(t)$.

(c) Compute the CTFT $X(j\omega)$.

(d) Compute the CTFT $H(j\omega)$.

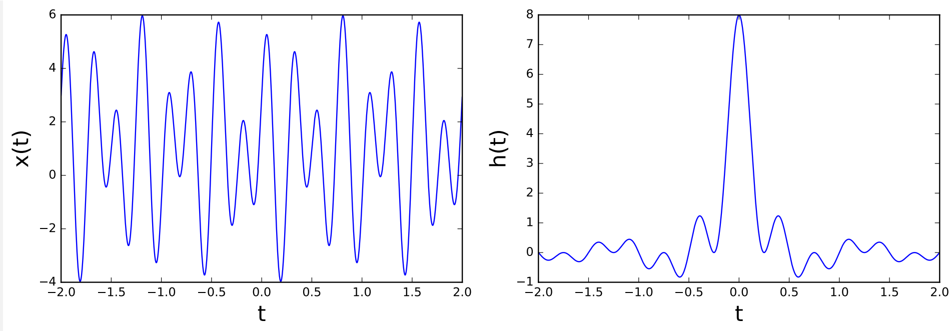
(e) Sketch $|H(j\omega)|$ over $-10\pi \le \omega \le 10\pi$. Verify your sketch using MATLAB.

(f) Find the output CTFT $Y(j\omega)$ and $y(t)$.

Try to explain in words what the system is doing.

Solution:

(a) Here's a sketch

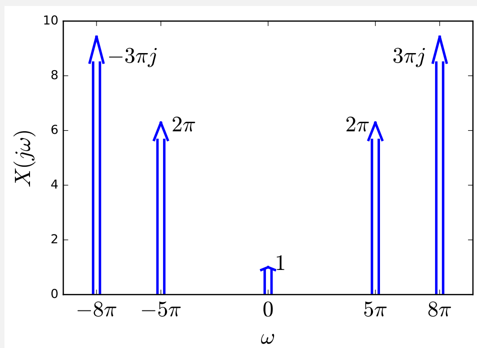


(b) The signal $x(t)$ is the sum of signals with period $\frac{2\pi}{5\pi} = \frac{2}{5}$ and $\frac{2\pi}{8\pi} = \frac{1}{4}$, so the fundamental period is $T = 2$, which corresponds to $\omega_0 = \frac{2\pi}{2} = \pi$.

(c) The CTFT is

$$X(j\omega) = \delta(\omega) + 2\pi\delta(\omega + 5\pi) + 2\pi\delta(\omega - 5\pi) + j3\pi\delta(\omega + 8\pi) - j3\pi\delta(\omega - 8\pi) \tag{95}$$

Here's a sketch:



(d) For $H(j\omega)$ we have a sinc multiplied by a cosine in time, which becomes a rect convolved with two delta functions in frequency:

$$\mathrm{sinc}(4t) \xleftrightarrow{\text{CTFT}} \frac{\pi}{4}\,\mathrm{rect}\left(\frac{\omega}{8}\right) \tag{96}$$

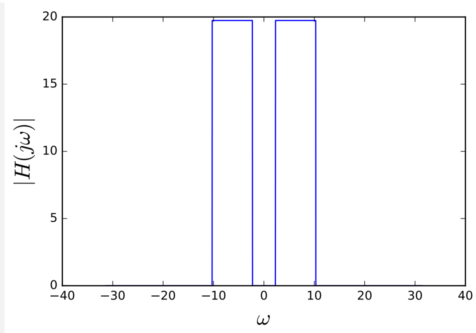
$$8\,\mathrm{sinc}(4t) \xleftrightarrow{\text{CTFT}} 2\pi\,\mathrm{rect}\left(\frac{\omega}{8}\right) \tag{97}$$

$$8\,\mathrm{sinc}(4t)\cos(4\pi t) \xleftrightarrow{\text{CTFT}} \frac{1}{2\pi}\left(2\pi\,\mathrm{rect}\left(\frac{\omega}{8}\right)\right) * \left(\pi\delta(\omega - 4\pi) + \pi\delta(\omega + 4\pi)\right). \tag{98}$$

The first term is a rectangle from -4 to 4 of height 2π and the second term is two delta functions of height π centered at $\pm 4\pi$. Since each delta function makes a copy of the rectangle centered at that delta function, we have

$$H(j\omega) = \pi\,\mathrm{rect}\left(\frac{\omega + 4\pi}{8}\right) + \pi\,\mathrm{rect}\left(\frac{\omega - 4\pi}{8}\right) \tag{99}$$

(e) Since $H(j\omega)$ is real, $|H(j\omega)| = H(j\omega)$. Here is a plot:



The box boundaries are at $-2\pi \pm 4$ and $2\pi \pm 4$.

(f) The output is just the two boxes times those delta functions. The delta functions at 0 and $\pm 8\pi$ are outside the boxes, leaving only the delta functions at $\pm 5\pi$. Those get a gain of $2\pi^2$, so

$$Y(j\omega) = 2\pi^2 \delta(\omega - 5\pi) + 2\pi^2 \delta(\omega + 5\pi) \tag{100}$$

In the time domain this is

$$y(t) = 2\pi \cos(5\pi t). \tag{101}$$

Problem 8 (ECE 345 Fall 2017 Midterm 2)**.** Consider the following signal $x(t)$:

$$x(t) = e^{-3|t|} \cos(\omega_c t). \tag{102}$$

where $\omega_c = 100\pi$.

(a) Calculate the CT Fourier transform $X(j\omega)$ of $x(t)$.

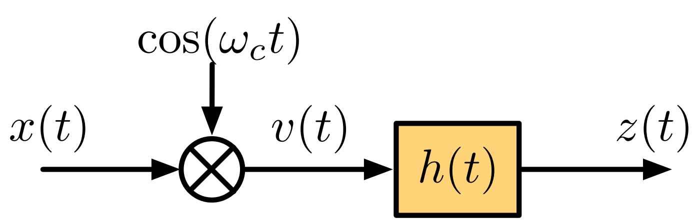
(b) Plot $X(j\omega)$.

(c) Suppose the signal $x(t)$ is passed through an LTI system with impulse response

$$h(t) = \frac{\sin(50\pi t)}{\pi t} \tag{103}$$

What is the output $y(t)$ of the LTI system? You can give your answer in the time or frequency domain.

(d) Instead, suppose you implement the following system, where $h(t)$ is the same LTI filter as (11):



Calculate $V(j\omega)$ and sketch it.

(e) Sketch the output $Y(j\omega)$.

Solution:

(a) Let $m(t) = e^{-3|t|}$. Then

$$M(j\omega) = \frac{6}{9+\omega^2}. \tag{104}$$

The CTFT of $\cos(100\pi t)$ is

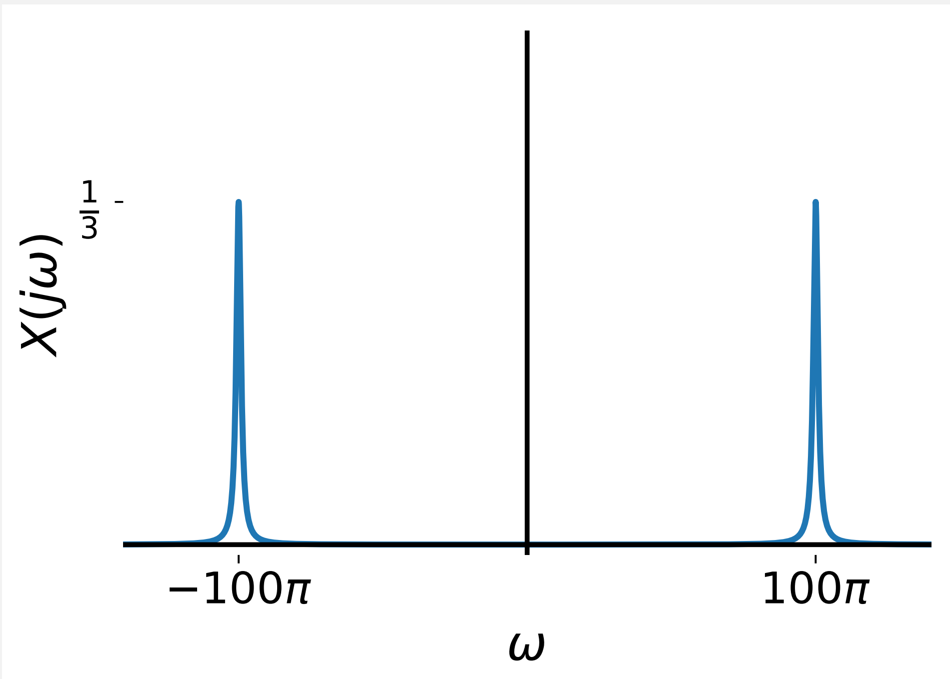
$$\pi\delta(\omega - 100\pi) + \pi\delta(\omega + 100\pi), \tag{105}$$

so

$$X(j\omega) = \frac{1}{2\pi}\left(\frac{6\pi}{9+(\omega-100\pi)^2} + \frac{6\pi}{9+(\omega+100\pi)^2}\right). \tag{106}$$

$$= \frac{3}{9+(\omega-100\pi)^2} + \frac{3}{9+(\omega+100\pi)^2}. \tag{107}$$

(b) Here is a plot:



(c) The system $h(t)$ is $(1/2)$ times an ideal lowpass filter from -50π to 50π. The output of the system is thus

$$Y(j\omega) = \begin{cases} \frac{3}{9+(\omega-100\pi)^2} + \frac{3}{9+(\omega+100\pi)^2}. & \omega \in [-50\pi, 50\pi] \\ 0 & \text{otherwise} \end{cases} \tag{108}$$

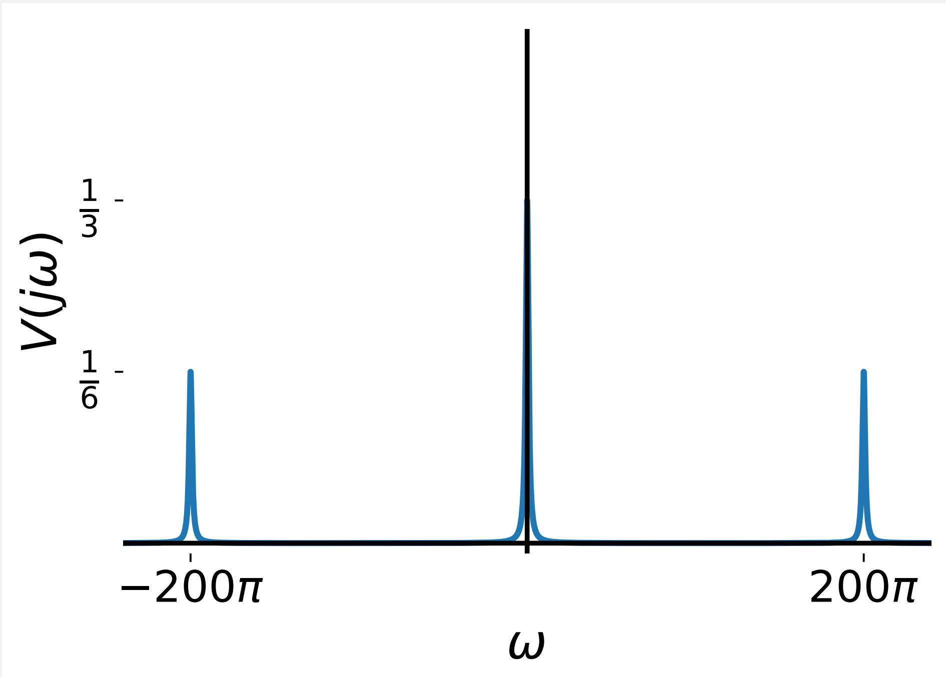
(d) Multiplying by another cosine is convolving with another

$$\pi\delta(\omega - 100\pi) + \pi\delta(\omega + 100\pi), \tag{109}$$

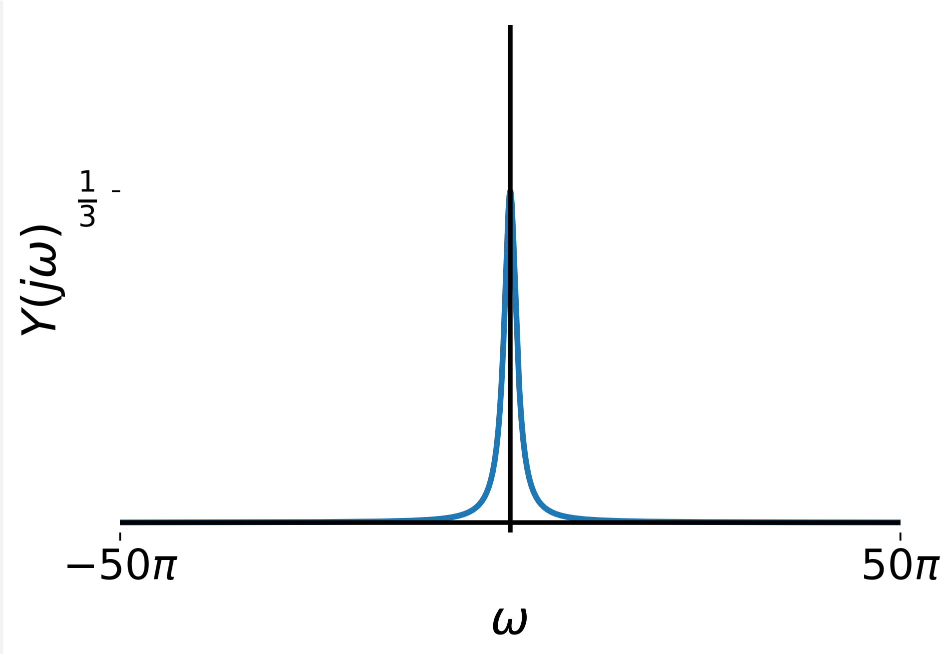
so

$$V(j\omega) = \frac{1}{2\pi}\left(\frac{3\pi}{9+(\omega-200\pi)^2} + \frac{6\pi}{9+\omega^2}. + \frac{3\pi}{9+(\omega+200\pi)^2}\right) \tag{110}$$

$$= \frac{3/2}{9+(\omega-200\pi)^2} + \frac{3}{9+\omega^2}. + \frac{3/2}{9+(\omega+200\pi)^2}. \tag{111}$$



(e) Passing $V(j\omega)$ through an ideal lowpass filter just cuts off everything away outside of $|\omega| < 50\pi$ so we have the following picture:



Problem 9 (Nyquiste Rate, OW 7.3)**.** Find the Nyquist rate of the following signals.

(a) $x(t) = 1 + \cos(1000\pi t) + \cos(3000\pi t)$

(b) $x(t) = \frac{\sin(4000\pi t)}{\pi t}$

Solution:

(a) The largest frequency is 3000π, so we need $\omega_s = 6000\pi$.

(b) In the frequency domain, this is rectangle from -4000π to 4000π so $\omega_s = 8000\pi$.

Problem 10 (Nyquiste Rate, OW 7.4(c))**.** If $x(t)$ has Nyquist rate ω_0, find the Nyquist rate of $y(t) = (x(t))^2$.

Solution:

If $x(t)$ has a Nyquist rate of ω_0 then it has a total bandwidth of ω_0. Now, when we square it, $Y(j\omega) = \frac{1}{2\pi}X(j\omega) * X(j\omega)$ so the convolution of $X(j\omega)$ with itself will have a total bandwidth of $2\omega_0$. So $X_s = 2\omega_0$.