ECE 345: Linear Systems and Signals
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Fall 2020
Due: Friday, November 20, 2020

Solutions for Homework 6

Problem 1 (Computing inverse DTFTs)**.** Compute the time domain signals corresponding to the following DTFTs.

(a) $X(e^{j\omega}) = 1$ for $\pi/4 \le |\omega| \le 3\pi/4$ and 0 elsewhere.

(b) $X(e^{j\omega}) = e^{-j\omega/2}$ for $-\pi \le \omega \le \pi$

(c) $X(e^{j\omega}) = \cos^2(\omega) + \sin^2(3\omega)$.

(d) $X(e^{j\omega}) = \frac{e^{-j\omega} - \frac{1}{5}}{1 - \frac{1}{5}e^{-j\omega}}$.

Solution:

(a) This is an ideal bandpass filter (BPF) (draw a picture). We can solve this a couple of different ways, but let's try direct integration first:

$$x[n] = \frac{1}{2\pi}\int_{-3\pi/4}^{-\pi/4} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\pi/4}^{3\pi/4} e^{j\omega n} d\omega = \frac{1}{\pi n}\left(\sin\left(\frac{3\pi}{4}n\right) - \sin\left(\frac{\pi}{4}n\right)\right). \tag{1}$$

Another way to think about this is that it's an ideal LPF with cutoff $3\pi/4$ minus an ideal LPF with cutoff $\pi/4$:

$$x[n] = \frac{\sin\left(\frac{3\pi}{4}n\right)}{\pi n} - \frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n} \tag{2}$$

Yet another way to do it would be to think of it as an 2 times an ideal LPF with cutoff $\pi/4$ modulated by a cosine at $\pi/2$.

The key to solving this one is to first sketch the spectrum (draw a picture) and then figure out how to build that picture from simpler things you know, like an ideal LPF.

(b) For this one you might be tempted to go the same route as the previous part, but then you would get $\delta[n - 1/2]$ which makes no sense since we're in a DT scenario. Instead, you have to go back to the definition/integral:

$$\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega/2} e^{j\omega n} d\omega && (3)\\
&= \frac{1}{2\pi}\left[\frac{1}{j(n-1/2)} e^{j\omega(n-1/2)}\right]_{\omega=-\pi}^{\pi} && (4)\\
&= \frac{1}{j2\pi(n-1/2)}\left(e^{j\pi n}e^{-j\pi/2} - e^{-j\pi n}e^{j\pi/2}\right) && (5)\\
&= \frac{1}{j2\pi(n-1/2)}\left(-j(-1)^n - j(-1)^n\right) && (6)\\
&= \frac{(-1)^{n+1}}{\pi(n-1/2)}. && (7)
\end{aligned}$$

The question you may ask yourself is "when do I know that I should use the definition/integral to do the problem?" This is a matter of taste, but a one tip-off is that the DTFT is pretty simple looking (algebraically) but none of the properties look like they apply "cleanly" to the problem.

(c) This one requires some trigonometry identities, or you can just use Euler:

$$\begin{aligned} X(e^{j\omega}) &= \left(\frac{1}{2}e^{j\omega} + \frac{1}{2}e^{-j\omega}\right)^2 + \left(\frac{1}{2j}e^{j3\omega} - \frac{1}{2j}e^{-j3\omega}\right)^2 && (8) \\ &= \frac{1}{4}e^{-j2\omega} + \frac{1}{2} + \frac{1}{4}e^{j2\omega} - \frac{1}{4}e^{-j6\omega} - \frac{1}{4}e^{j6\omega} + \frac{1}{2} && (9) \\ &= 1 - \frac{1}{4}e^{-j6\omega} + \frac{1}{4}e^{-j2\omega} + \frac{1}{4}e^{j2\omega} - \frac{1}{4}e^{j6\omega} && (10) \end{aligned}$$

So now we can use the previous trick and read these off as a sum of delta functions:

$$x[n] = -\frac{1}{4}\delta[n-6] + \frac{1}{4}\delta[n-2] + \delta[n] + \frac{1}{4}\delta[n+2] - \frac{1}{4}\delta[n+6]. \tag{11}$$

(d) We can see this one more easily in the z-domain:

$$\begin{aligned} X(z) &= \frac{z^{-1} - 1/5}{1 - \frac{1}{5}z^{-1}} && (12) \\ x[n] &= \left(\frac{1}{5}\right)^{n-1} u[n-1] - \frac{1}{5}\left(\frac{1}{5}\right)^{n} u[n] && (13) \\ &= \left(\frac{1}{5}\right)^{n-1} u[n-1] - \left(\frac{1}{5}\right)^{n+1} u[n] && (14) \end{aligned}$$

Problem 2 (Using DTFT properties)**.** Suppose $x[n] \overset{\text{DTFT}}{\longleftrightarrow} X(e^{j\omega})$. Use DTFT properties to find the DTFTs of the following signals

(a) $x_1[n] = x[1-n] + x[-1-n]$

(b) $x_3[n] = (n-1)^2 x[n]$

Solution:

(a) We can build this up:

$$\begin{aligned} x[-n] &\overset{\text{DTFT}}{\longleftrightarrow} X(e^{-j\omega}) && (15) \\ x[-n+1] = x[-(n-1)] &\overset{\text{DTFT}}{\longleftrightarrow} e^{-j\omega}X(e^{-j\omega}) && (16) \\ x[-n-1] = x[-(n+1)] &\overset{\text{DTFT}}{\longleftrightarrow} e^{j\omega}X(e^{-j\omega}) && (17) \\ x_1[n] &\overset{\text{DTFT}}{\longleftrightarrow} 2X(e^{-j\omega})\cos(\omega). && (18) \end{aligned}$$

(b) We can expand the quadratic and then use the differentiation in frequency property:

$$x_3[n] = (n^2 - 2n + 1)x[n] \tag{19}$$

$$nx[n] \xleftrightarrow{\text{DTFT}} j\frac{d}{d\omega}X(e^{j\omega}) \tag{20}$$

$$n^2x[n] \xleftrightarrow{\text{DTFT}} j^2\frac{d}{d\omega}\frac{d}{d\omega}X(e^{j\omega}) = -\frac{d^2}{d\omega^2}X(e^{j\omega}) \tag{21}$$

$$x_3[n] \xleftrightarrow{\text{DTFT}} (-\frac{d^2}{d\omega^2} - 2j\frac{d}{d\omega} + 1)X(e^{j\omega}). \tag{22}$$

Problem 3 (A mystery signal!)**.** Suppose you know the following four facts about a real signal $x[n]$ with DTFT $X(e^{j\omega})$:

1. $x[n] = 0$ for $n > 0$.
2. $x[0] > 0$
3. $\mathfrak{Im}\{X(e^{j\omega})\} = \sin(\omega) - \sin(2\omega)$
4. $\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 3$

Determine $x[n]$. *Hint: You may want to use Parseval's relation.*

Solution:

The last fact is from Parseval's theorem, and it says that

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = 3 \tag{23}$$

$$\tag{24}$$

Since $x[n] = 0$ for positive n,

$$\sum_{n=-\infty}^{0} x[n]^2 = 3. \tag{25}$$

The third fact is a bit puzzling, but since the signal is real, we can think of its DTFT as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{0} x[n]e^{-j\omega n} \tag{26}$$

$$= \sum_{n=0}^{\infty} x[-n]e^{j\omega n} \tag{27}$$

$$= x[0] + \sum_{n=1}^{\infty} x[-n](\cos(\omega n) + j\sin(\omega n)) \tag{28}$$

That means

$$\mathfrak{Im}\{X(e^{j\omega})\} = \sum_{n=0}^{\infty} x[-n]\sin(\omega n) \tag{29}$$

Since the only two nonzero terms are for $n = 1$ and $n = 2$, we can see that $x[-1] > 0$, $x[-2] > 0$ and $x[n] = 0$ for all other values. Furthermore, $x[-1] = 1$ and $x[-2] = -1$.

Now we just need to find $x[0]$. Using Parseval, we can see that $x[0]^2 = 3 - x[-1]^2 - x[-2]^2 = 1$. Since $x[0] > 0$, we must have $x[0] = 1$. Putting it all together:

$$x[n] = \delta[n] + \delta[n+1] - \delta[n+2]. \tag{30}$$

Problem 4 (Filtering)**.** Sincs and rects are important in DT signal processing as well.

(a) Find the DTFT of $h[n] = \frac{\sin(Wn)}{\pi n}$ and sketch it.

(b) Suppose

$$x[n] = \sin\left(\frac{\pi n}{8}\right) - 2\cos\left(\frac{\pi n}{4}\right) \tag{31}$$

Find the DTFT of $x[n]$.

(c) Find the output of the following systems with input $x[n]$:

$$h_1[n] = \frac{\sin(\pi n/6)}{\pi n} \tag{32}$$

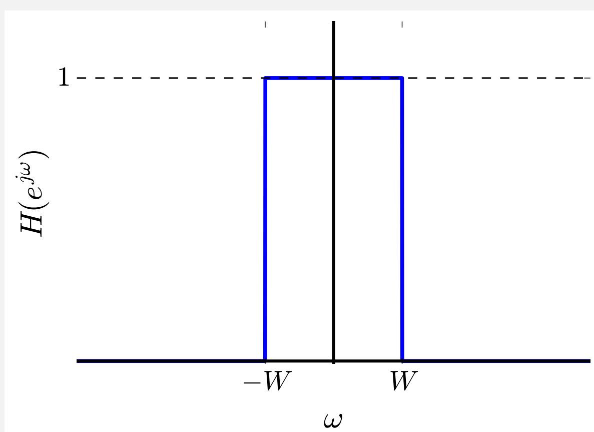
$$h_2[n] = \frac{\sin(\pi n/6)}{\pi n} + \frac{\sin(\pi n/2)}{\pi n} \tag{33}$$

$$h_3[n] = \frac{\sin(\pi n/6)\sin(\pi n/3)}{\pi^2 n^2} \tag{34}$$

$$h_4[n] = \frac{\sin(\pi n/6)\sin(\pi n/3)}{\pi n} \tag{35}$$

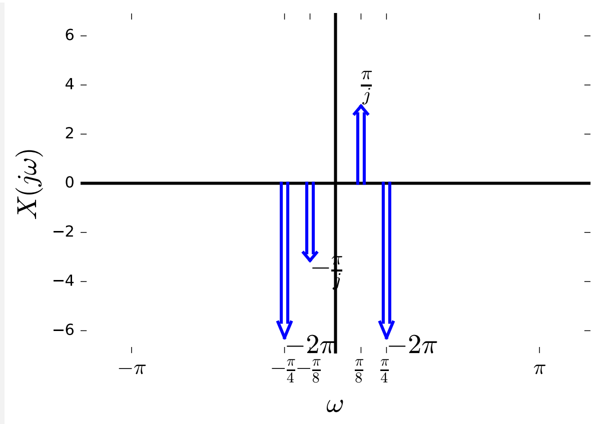
Solution:

(a) This is an ideal LPF with cutoff W, so it looks like this between $[-\pi, \pi]$:



(b) The DTFT of $x[n]$ in $[-\pi, \pi]$ is:

$$X(e^{j\omega}) = \frac{\pi}{j}\delta(\omega - \pi/8) - \frac{\pi}{j}\delta(\omega + \pi/8) - 2\pi\delta(\omega - \pi/4) - 2\pi\delta(\omega + \pi/4). \tag{36}$$



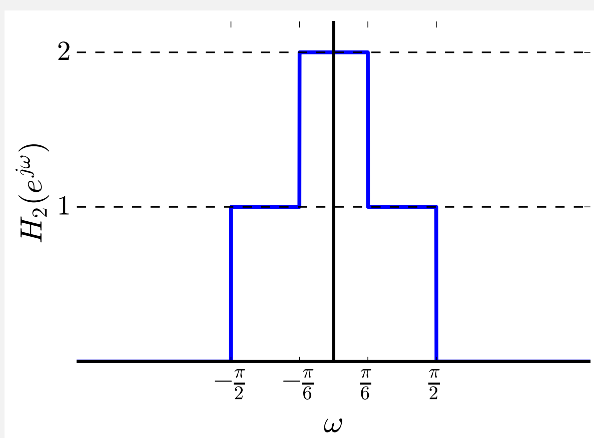
(c) For each of these filters, we have to figure out what the frequency response is and then use those to weigh the delta functions in $X(e^{j\omega})$.

The filter $h_1[n]$ is an ideal LPF with cutoff $\pi/6$ so only components at $|\omega| < \pi/6$ will make it through the filter and they will each get gain 1. That means between $[-\pi, \pi]$

$$Y_1(e^{j\omega}) = \frac{\pi}{j}\delta(\omega - \pi/8) - \frac{\pi}{j}\delta(\omega + \pi/8) \tag{37}$$

$$y_1[n] = \sin\left(\frac{\pi n}{8}\right). \tag{38}$$

The filter $h_2[n]$ is the sum of two ideal LPFs with cutoffs $\pi/6$ and $\pi/2$. That means that, for $-\pi \le \omega \le \pi$, the gain for components with $|\omega| < \pi/6$ is 2 and the gain for components in $\pi/6 < |\omega| < \pi/2$ is 1:



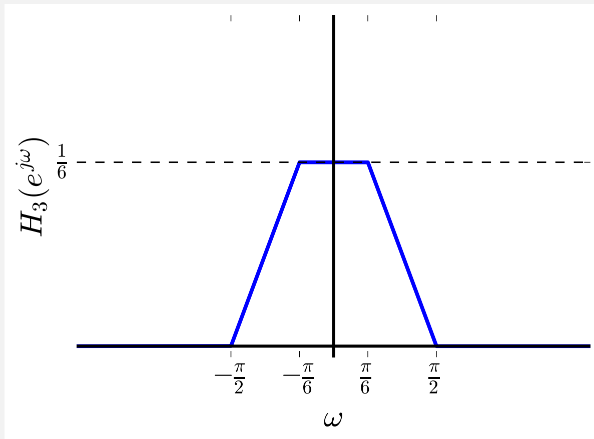
Therefore the sine component gets a gain of 2 and the cosine a gain of 1:

$$y_2[n] = 2\sin\left(\frac{\pi n}{8}\right) - 2\cos\left(\frac{\pi n}{4}\right) \tag{39}$$

The filter $h_3[n]$ is the product in time of ideal LPFs with cutoffs $\pi/3$ and $\pi/6$. The corresponding spectrum is the periodic convolution of the two filters:

$$H_3(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{rect}\left(\frac{\theta}{\pi/3}\right)\mathrm{rect}\left(\frac{\omega - \theta}{2\pi/3}\right)d\theta \tag{40}$$

Don't forget the $\frac{1}{2\pi}$! Since the sum of the bandwidths of the two filters is $\pi < 2\pi$ there is no aliasing and we have the following graphical convolution of the two filters for ω in $[-\pi, \pi]$:



We just have to read off the gains for the different delta functions in $X(e^{j\omega})$. At $\pm\pi/8$ the gain is 1/6. At $\pm\pi/4$ the gain is 1/8. Therefore

$$y_3[n] = \frac{1}{6}\sin\left(\frac{\pi n}{8}\right) - \frac{1}{4}\cos\left(\frac{\pi n}{4}\right) \tag{41}$$

Finally, for $h_4[n]$, since there is no n^2 in the denominator, this is just a sinc times a sine, which modulates a LPF with cutoff $\pi/6$ to be centered at $\pm\pi/3$, so for ω in $[-\pi, \pi]$:

$$H_4(e^{j\omega}) = \begin{cases} -\frac{1}{2j} & -\pi/2 < \omega < -\pi/6 \\ \frac{1}{2j} & \pi/6 < \omega < \pi/2 \\ 0 & \text{otherwise} \end{cases} \tag{42}$$

Now the sine gets gain 0 and the cosine components get complex gains, so for ω in $[-\pi, \pi]$:

$$Y(e^{j\omega}) = \frac{-2\pi}{-2j}\delta(\omega + \pi/4) - \frac{2\pi}{2j}\delta(\omega - \pi/4) \tag{43}$$

$$= \frac{\pi}{j}\delta(\omega + \pi/4) - \frac{\pi}{j}\delta(\omega - \pi/4) \tag{44}$$

$$y_4[n] = -\sin(\pi n/4). \tag{45}$$

This shows that filters can cause phase shifts such that a signal entering as a cosine exits as a sine.

Problem 5. Using the MATLAB function `freqz`, compute and plot the real and imaginary parts as well as magnitude and phase functions of the following DTFTs:

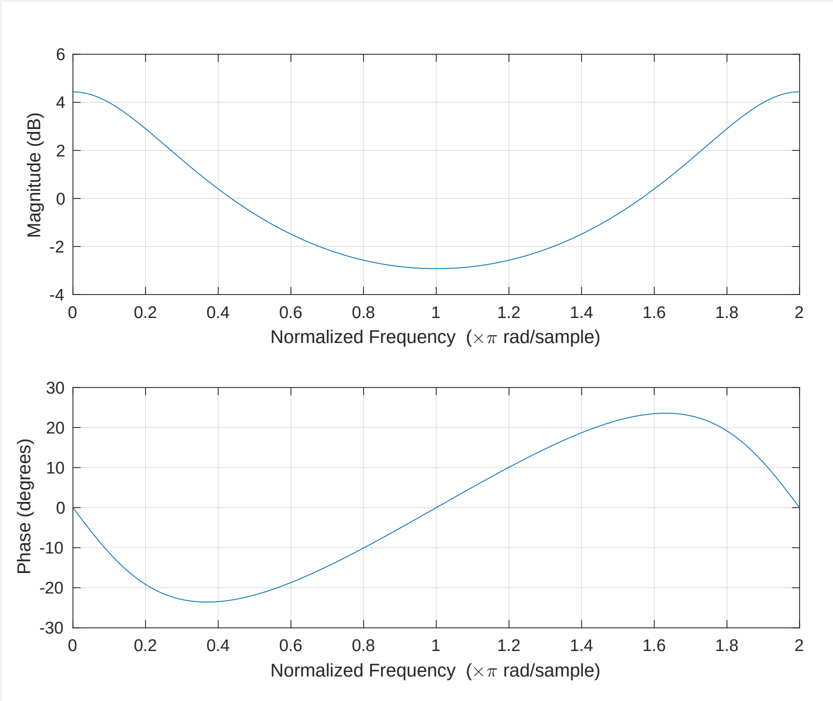
$$X(e^{j\omega}) = \frac{1}{1 - 0.4e^{-j\omega}} \tag{46}$$

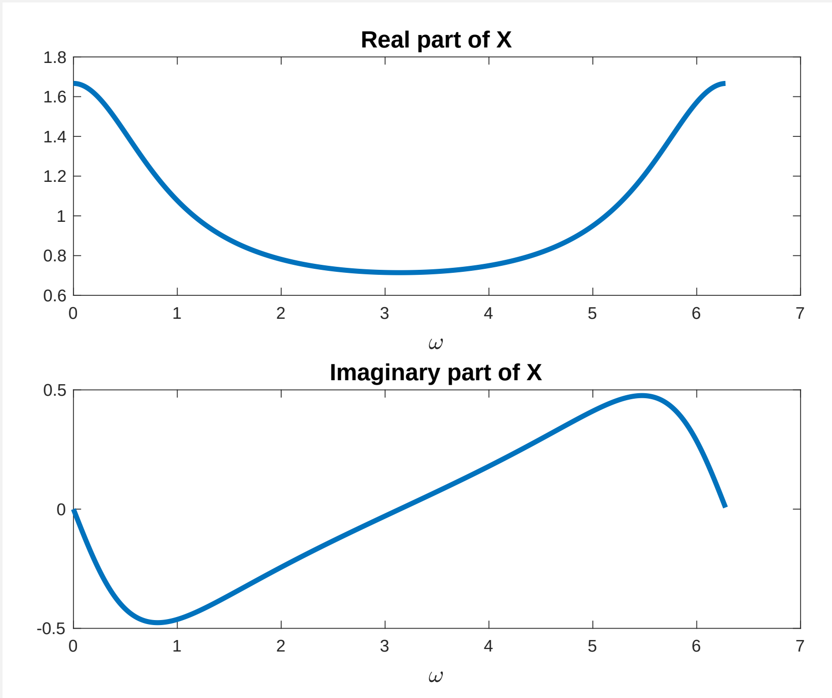
$$Y(e^{j\omega}) = \frac{0.2 + 0.4e^{-j\omega} + e^{-2j\omega}}{1 + 0.4e^{-j\omega} + 0.2e^{-2j\omega}} \tag{47}$$

Make sure to use appropriate frequency points while computing and plotting the DTFTs. Also, use the MATLAB function `unwrap` to unwrap the phase functions.

Solution:

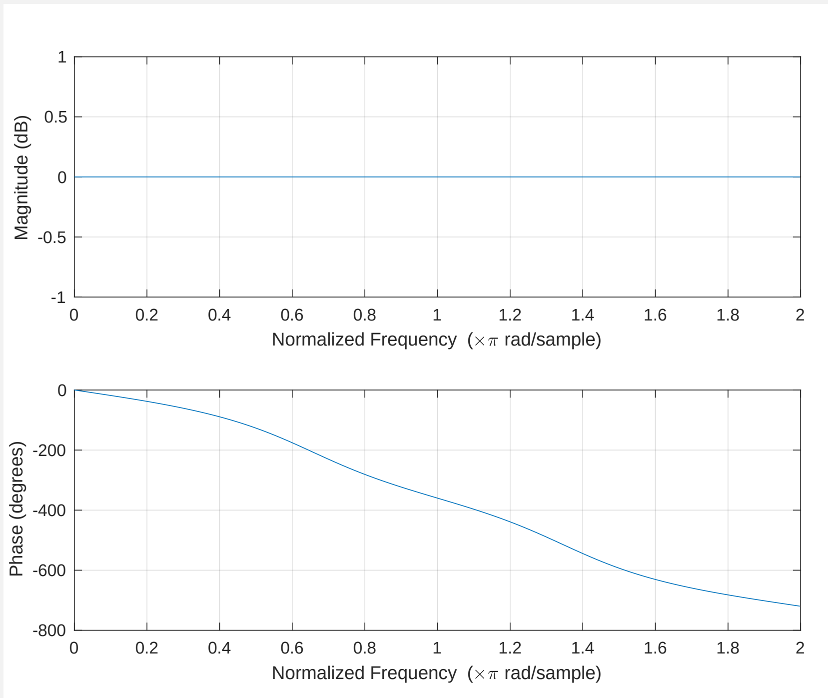
Here are the plots for $X(e^{j\omega})$:

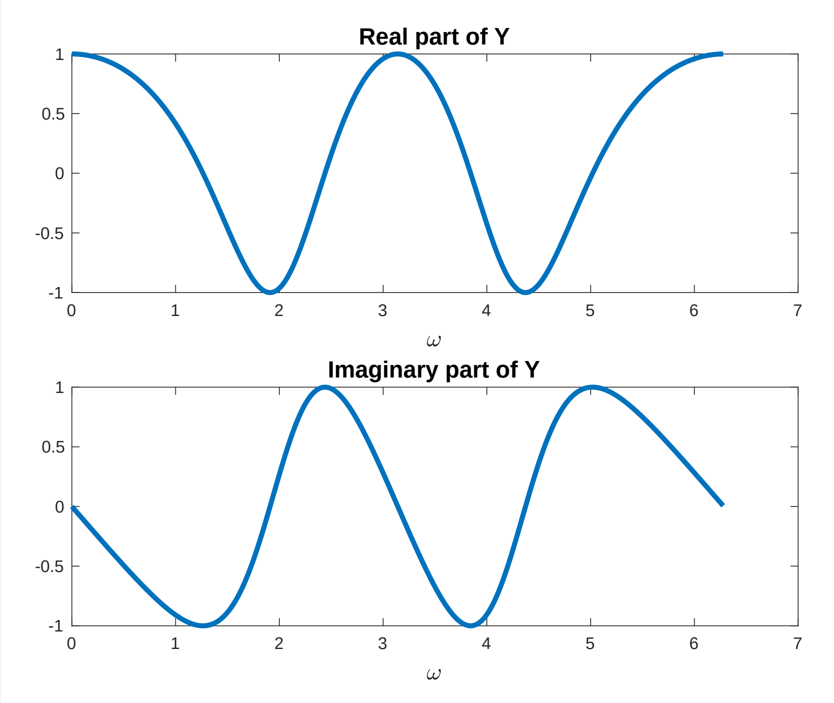

Magnitude (dB)
Normalized Frequency (×π rad/sample)
Phase (degrees)
Normalized Frequency (×π rad/sample)



From this we can see that X has a sort of highpass characteristic (since it is large around π).

Here are the plots for $Y(e^{j\omega})$:


Magnitude (dB)
Normalized Frequency (×π rad/sample)
Phase (degrees)
Normalized Frequency (×π rad/sample)



The filter Y has a very interesting property – the magnitude is 0 dB for all frequencies! This is a special sort of system called an *allpass* system – all frequencies get the same magnitude gain, but as you can see, there are different phase shifts at different frequencies.