ECE 345: Linear Systems and Signals
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Due: 5 PM, November 6, 2020

Solutions for Homework 5

Problem 1 (Oppenheim, Willsky, and Nawab, 10.8)**.** Let $x[n]$ be a signal whose rational z-transform $X(z)$ contains a pole at $z = 1/2$. Given that

$$x_1[n] = \left(\frac{1}{4}\right)^n x[n] \tag{1}$$

is absolutely summable and

$$x_2[n] = \left(\frac{1}{8}\right)^n x[n] \tag{2}$$

is absolutely summable, determine whether $x[n]$ is left-sided, right-sided, or two-sided.

Solution:

This is a (short) exercise in marshaling facts about z-transforms to solve a mystery. Since there is a pole at $z = 1/2$ and $X(z)$ is rational, we know that

$$X(z) = \frac{P(z)}{Q(z)(1 - \frac{1}{2}z^{-1})}. \tag{3}$$

the ROC is either outside the circle $|z| = 1/2$ or it is inside the circle $|z| = 1/2$.

What we get from the two facts is that $X_1(z)$ and $X_2(z)$ both have ROCs that contain the unit circle. This is because absolutely summable implies that corresponding LTI system is stable, which means the ROC has to contain the unit circle.

If $X(z)$ were left-sided, then $x[n]$ would contain a term like $-\left(\frac{1}{2}\right)^n u[-n-1]$, which is increasing exponentially fast going to the left (as $n \to -\infty$). This term multiplied by $\left(\frac{1}{4}\right)^n$ cannot correspond to a summable transform, so $X(z)$ can't be left-sided.

If $X(z)$ were two-sided, then the term corresponding to $z = 1/2$ could be right-sided, in which case we need to check if there are left-sided signals such that $x_1[n]$ and $x_2[n]$ are summable. This can happen as long as the left-sided signal is decaying as $n \to \infty$ – this corresponds to a pole outside the unit circle, and one that is sufficiently large so that $\alpha^n \left(\frac{1}{4}\right)^n$ and $\alpha^n \left(\frac{1}{8}\right)^n$ both decay as $n \to -\infty$. This can happen when $\alpha > 8$, so it is possible for $X(z)$ to be two-sided.

Similarly, $X(z)$ could also be right-sided, using the same argument. In this case the ROC could be $|z| > \alpha$, for example. What if $x_2[n]$ were not absolutely summable? Then we know that the α cannot be > 8 so there is a pole outside the unit circle and therefore $x[n]$ is two-sided.

Problem 2 (Oppenheim, Willsky, and Nawab 10.23)**.** Determine the inverse z-transform for the following signals

(a) $X(z) = \frac{1-z^{-1}}{1-\frac{1}{4}z^{-2}}$, with ROC $|z| < \frac{1}{2}$

(b) $X(z) = \frac{z^{-1}-\frac{1}{2}}{1-\frac{1}{2}z^{-1}}$, with ROC $|z| < \frac{1}{2}$

(c) $X(z) = \frac{z^{-1}-\frac{1}{2}}{(1-\frac{1}{2}z^{-1})^2}$, with ROC $|z| < \frac{1}{2}$

Solution:

Note these are all left-sided transforms since the ROC is inside a circle.

The key to the first problems is to do a partial fraction expansion and then take the inverse.

(a) For the first one, there are poles at $\pm\frac{1}{2}$. Doing the partial fraction expansion:

$$X(z) = \frac{-\frac{1}{2}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{3}{2}}{1+\frac{1}{2}z^{-1}} \tag{4}$$

Taking left-sided inverses,

$$x[n] = \frac{1}{2}\left(\frac{1}{2}\right)^n u[-n-1] - \frac{3}{2}\left(-\frac{1}{2}\right)^n u[-n-1]. \tag{5}$$

(b) For the second one, we have equal degree in the numerator and denominator so we have to factor out a constant term in the partial fraction expansion. In the end we get

$$X(z) = -2 + \frac{\frac{3}{2}}{1-\frac{1}{2}z^{-1}} \tag{6}$$

$$x[n] = -2\delta[n] - \frac{3}{2}\left(\frac{1}{2}\right)^n u[-n-1] \tag{7}$$

(c) For the last one we may need to use a transform from the table:

$$\frac{az^{-1}}{(1-az^{-1})^2} \overset{\mathcal{Z}}{\Longleftrightarrow} -na^n u[-n-1] \tag{8}$$

So plugging in $a = \frac{1}{2}$:

$$\frac{z^{-1}}{(1-\frac{1}{2}z^{-1})^2} \overset{\mathcal{Z}}{\Longleftrightarrow} -2n\left(\frac{1}{2}\right)^n u[-n-1]. \tag{9}$$

Using the fact that multiplication by z is advancing by 1:

$$\frac{-\frac{1}{2}}{(1-\frac{1}{2}z^{-1})^2} \overset{\mathcal{Z}}{\Longleftrightarrow} n\left(\frac{1}{2}\right)^{n+1} u[-n-2] \tag{10}$$

Then

$$x[n] = -2n\left(\frac{1}{2}\right)^n u[-n-1] + (n+1)\left(\frac{1}{2}\right)^{n+1} u[-n-2] \tag{11}$$

Problem 3 (SSTA 7.24)**.** A causal LTI system has zeros at $\{3, 4\}$ and poles at $\{1, 2\}$. The transfer function $H(z)$ has $H(0) = 6$. Compute:

(a) The transfer function H(z).

(b) The response to $x[n] = \delta[n] - 3\delta[n-1] + 2\delta[n-2]$.

(c) The impulse response $h[n]$.

(d) The difference equation.

Solution:

(a) We can write out $H(z)$ as a rational transform using the poles and zeros:

$$H(z) = C\frac{(z-3)(z-4)}{(z-1)(z-2)} = C\frac{(1-3z^{-1})(1-4z^{-1})}{(1-z^{-1})(1-2z^{-1})} \tag{12}$$

The second way of writing things is how we have done z-transforms in the lecture, this emphasizes the difference equation form of the transfer function.

We need to figure out what the scaling constant C is. To do this we use our last piece of evidence: $H(0) = 6$. Note that to use this evidence it is better to have the transfer function in terms of z rather in terms of z^{-1}. Plugging in:

$$H(0) = C\frac{12}{2} = 6C, \tag{13}$$

so $C = 1$. Therefore

$$H(z) = \frac{z^2 - 7z + 12}{z^2 - 3z + 2}. = \frac{1 - 7z^{-1} + 12z^{-2}}{1 - 3z^{-1} + 2z^{-2}}. \tag{14}$$

(b) To compute the response we use the fact that convolution in time is multiplication in the z-domain:

$$x[n] = \delta[n] - 3\delta[n-1] + 2\delta[n-2] \tag{15}$$
$$X(z) = 1 - 3z^{-1} + 2z^{-2} \tag{16}$$

Conveniently, this is the same as the denominator of $H(z)$:

$$Y(z) = H(z)X(z) = 1 - 7z^{-1} + 12z^{-2} \tag{17}$$
$$y[n] = \delta[n] - 7\delta[n-1] + 12\delta[n-2]. \tag{18}$$

This emphasizes again why (sometimes) writing $H(z)$ as a polynomial in z^{-1} is helpful: you can just "read off" $y[n]$ by translating az^{-k} to $a\delta[n-k]$.

(c) We need to do partial fraction expansion on $H(z)$. Since the degree is the same on the numerator and denominator we need to first divide out a constant term:

$$H(z) = 1 + \frac{(1 - 7z^{-1} + 12z^{-2}) - (1 - 3z^{-1} + 2z^{-2})}{1 - 3z^{-1} + 2z^{-2}} \tag{19}$$
$$= 1 + z^{-1}\frac{-4 + 10z^{-1}}{(1-z^{-1})(1-2z^{-1})}. \tag{20}$$

To go back to the time-domain, the first term will map to $\delta[n]$ and we can interpret the second term as the inverse transform of $\frac{-4+10z^{-1}}{(1-z^{-1})(1-2z^{-1})}$ delayed by 1, since multiplication by z^{-1} is the same as a delay by 1. So for our regular partial fraction expansion:

$$\left.\frac{-4 + 10z^{-1}}{(1-z^{-1})(1-2z^{-1})}(1-z^{-1})\right|_{z=1} = \frac{-4+10}{-1} = -6 \tag{21}$$
$$\left.\frac{-4 + 10z^{-1}}{(1-z^{-1})(1-2z^{-1})}(1-2z^{-1})\right|_{z=2} = \frac{-4+5}{1/2} = 2 \tag{22}$$

Putting it all together,

$$H(z) = 1 + z^{-1}\left(\frac{2}{1-2z^{-1}} - \frac{6}{1-z^{-1}}\right) \tag{23}$$

$$h[n] = \delta[n] + \text{delay}_1\{2(2)^n u[n] + 6u[n]\} \tag{24}$$

$$= \delta[n] + 2(2)^{n-1}u[n-1] - 6u[n-1]. \tag{25}$$

Here we've made up some notation $\text{delay}_k\{x[n]\} = x[n-k]$ to emphasize the *operational significance* of multiplying by z^{-1}.

(d) We can read the difference equation right off of the formula for $H(z)$ in terms of z^{-1}:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-7z^{-1}+12z^{-2}}{1-3z^{-1}+2z^{-2}} \tag{26}$$

$$Y(z)(1-3z^{-1}+2z^{-2}) = X(z)(1-7z^{-1}+12z^{-2}) \tag{27}$$

$$y[n] - 3y[n-1] + 2y[n-2] = x[n] - 7x[n-1] + 12x[n-2]. \tag{28}$$

Problem 4 (Final Exam, Fall 2017)**.** When the input to an LTI system is

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n-1] \tag{29}$$

the output is

$$y[n] = 6\left(\frac{1}{2}\right)^n u[n] - 6\left(\frac{3}{4}\right)^n u[n]. \tag{30}$$

(a) Find the Z-transforms of $X(z)$ and $Y(z)$. Write them as ratios of factorized polynomials in z^{-1}.

(b) Find the system transfer function $H(z)$. Plot the poles and zeros and indicate the region of convergence.

(c) Find the impulse response $h[n]$ of the system.

(d) Is $H(z)$ stable? Is it causal? Explain why or why not.

(e) Find the inverse system $H^{-1}(z)$. Can you find an ROC so the inverse is stable? For that choice is the inverse system causal? Explain why or why not.

Solution:

(a) First write the transforms of $x[n]$ and $y[n]$ with their possible ROCs:

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-2z^{-1}} \qquad \frac{1}{2} \le |z| \le 2 \tag{31}$$

$$= \frac{-\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})} \tag{32}$$

$$Y(z) = \frac{6}{1-\frac{1}{2}z^{-1}} - \frac{6}{1-\frac{3}{4}z^{-1}} \qquad |z| > \frac{3}{4} \tag{33}$$

$$= 6\frac{-\frac{1}{4}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{3}{4}z^{-1})} \tag{34}$$

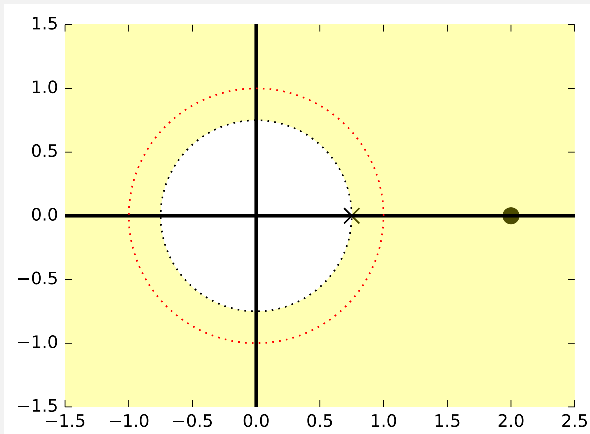
$$= \frac{-\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{3}{4}z^{-1})} \tag{35}$$

Note that the actual ROC of $Y(z)$ will be the intersection of the ROCs of $X(z)$ and $H(z)$, but as a signal on its own the ROC is as given.

(b) For the system, we divide the two:

$$H(z) = \frac{Y(z)}{X(z)} \tag{36}$$
$$= \frac{1 - 2z^{-1}}{1 - \frac{3}{4}z^{-1}}. \tag{37}$$

The pole-zero diagram is here:



For the ROC, we have a single pole at $\frac{3}{4}$ and a zero at 2. Since the ROC for $Y(z)$ is the intersection of the ROCs for $X(z)$ and $H(z)$, we have $|z| > \frac{3}{4}$ for the ROC of $H(z)$.

(c) For the impulse response, we have $\left(\frac{3}{4}\right)^n u[n]$ minus 2 times a unit delay of the same signal, so

$$h[n] = \left(\frac{3}{4}\right)^n u[n] - 2\left(\frac{3}{4}\right)^{n-1} u[n-1]. \tag{38}$$

(d) The transfer function has all poles inside the unit circle and the ROC extends from the largest pole to ∞ so it is causal and stable.

(e) The inverse system is

$$H^{-1}(z) = \frac{X(z)}{Y(z)} \tag{39}$$
$$= \frac{1 - \frac{3}{4}z^{-1}}{1 - 2z^{-1}}. \tag{40}$$

This has a pole at $z = 2$ and a zero at $z = \frac{3}{4}$. There are two possible ROCs: $|z| > 2$ and $|z| < 2$. If we choose the ROC $|z| < 2$ then the ROC contains the unit circle so it is stable. However, for this choice of ROC the inverse is not causal since the ROC goes inward rather than out to ∞.

Problem 5 (SSTA 7.27)**.** We observe the two signals which start at $n = 0$:

$$y_1[n] = (1, -13, 86, -322, 693, -945, 500) \tag{41}$$
$$y_2[n] = (1, -13, 88, -338, 777, -1105, 750), \tag{42}$$

where $y_1[n] = h_1[n] * x[n]$, $y_2[n] = h_2[n] * x[n]$, and all of $\{h_1[n], h_2[n], x[n]\}$ are unknown! We know only that all of the signals have finite lengths, are causal, and $x[0] = 1$. Compute $h_1[n]$, $h_2[n]$, and $x[n]$. Hint: Use z-transforms and zeros.

Note: this is a real application in communications. The signals $y_1[n]$ and $y_2[n]$ represent signals at a wireless receiver which has two antennas. The transmitted signal $x[n]$ is the same for both but the *channels* are different to each received antenna: these are the systems $h_1[n]$ and $h_2[n]$. The receiver may not know the channels (since they very in time) and must estimate the transmitted signals and the channels. This problem is called *blind equalization* (since the receiver is trying to equalize/normalize the channel without knowing anything) or *blind deconvolution.*

Use MATLAB if you need to do the calculations.

Solution:

Taking z-transforms of $y_1[n]$ and $y_2[n]$ gives

$$Y_1(z) = H_1(z)X(z) \tag{43}$$
$$Y_2(z) = H_2(z)X(z) \tag{44}$$

The zeros that Y_1 and Y_2 have in common must be the zeros of $X(z)$, whereas the zeros they have that are different are the zeros of H_1 and H_2.

So what are the zeros of $Y_1(z)$ and $Y_2(z)$? Well, we can look at them as polynomials:

$$Y_1(z) = 1 - 13z{-}1 + 86z{-}2, -322z{-}3 + 693z{-}4 - 945z{-}5 + 500z{-}6 \tag{45}$$
$$Y_2(z) = 1 - 13z{-}1 + 88z{-}2, -338z{-}3 + 777z{-}4 - 1105z{-}5 + 750z{-}6 \tag{46}$$

So each of them will have 6 zeros. Breaking out the trusty MATLAB, we can find the zeros:

$$Y_1(z) = 0 \Longrightarrow z \in \{1 \pm 2j, 3 \pm 4j, 1, 4\} \tag{47}$$
$$Y_2(z) = 0 \Longrightarrow z \in \{1 \pm 2j, 3 \pm 4j, 2, 3\} \tag{48}$$
$$\tag{49}$$

So the zeros they have in common are $\{1 \pm 2j, 3 \pm 4j\}$ – these belong to $X(z)$. That means

$$X(z) = 1 - 8z^{-1} + 42z^{-2} - 80z^{-3} + 125z^{-4} \tag{50}$$
$$x[n] = \delta[n] - 8\delta[n-1] + 42\delta[n-2] - 80\delta[n-3] + 125\delta[n-4] \tag{51}$$
$$\tag{52}$$

The remaining zeros that are not in common belong to $H_1(z)$ and $H_2(z)$, respectively:

$$H_1(z) = (1 - z^{-1})(1 - 4z^{-1}) = 1 - 5z^{-1} + 4z^{-2} \tag{53}$$
$$h_1[n] = \delta[n] - 5\delta[n-1] + 4\delta[n-2] \tag{54}$$
$$H_2(z) = (1 - 2z^{-1})(1 - 3z^{-1}) = 1 - 5z^{-1} + 6z^{-2} \tag{55}$$
$$h_2[n] = \delta[n] - 5\delta[n-1] + 6\delta[n-2]. \tag{56}$$

Note that we need the fact here that $x[0] = 1$, otherwise we would only know the solution up to a scaling factor.

Problem 6 (Oppenheim, Willsky, and Nawab, 5.21)**.** Compute the DTFTs of the following signals.

(a) $x[n] = u[n-2] - u[n-6]$

(b) $x[n] = \left(\frac{1}{2}\right)^{|n|} \cos\left(\frac{\pi}{8}(n-1)\right)$

(c) $x[n] = n$ for $|n| \leq 3$ and 0 elsewhere

(d) $x[n] = \sin(\frac{\pi}{2}n) + \cos(n)$

Solution:

(a) We can do this using direct calculation from the definition:

$$X(e^{j\omega}) = e^{-j2\omega} + e^{-j3\omega} + e^{-j4\omega} + e^{-j5\omega} \tag{57}$$

$$= e^{-j3.5\omega}\left(e^{j1.5\omega} + e^{j0.5\omega} + e^{-j0.5\omega} + e^{-j1.5\omega}\right) \tag{58}$$

$$= e^{-j3.5\omega}\left(2\cos(0.5\omega) + 2\cos(1.5\omega)\right). \tag{59}$$

Another approach is to factor out $e^{-j2\omega}$:

$$X(e^{j\omega}) = e^{-j2\omega}\sum_{k=0}^{3} e^{-j\omega k} \tag{60}$$

$$= e^{-j2\omega}\frac{1 - e^{-j4\omega}}{1 - e^{j\omega}} \tag{61}$$

$$= e^{-j3.5\omega}\frac{e^{j2\omega} - e^{-j2\omega}}{e^{j0.5\omega} - e^{-j0.5\omega}} \tag{62}$$

$$= e^{-j3.5\omega}\frac{\sin(2\omega)}{\sin(\omega/2)}. \tag{63}$$

(b) The trick to this one is that we can use the definition and then use Euler's formula on the cosine to pull it into the exponent. First let's split the double-sided $x[n]$ into two one-sided terms:

$$x[n] = \left(\frac{1}{2}\right)^n \cos\left(\frac{\pi}{8}(n-1)\right) u[n] + \left(\frac{1}{2}\right)^{-n} \cos\left(\frac{\pi}{8}(n-1)\right) u[-n-1]. \tag{64}$$

We can take the transform of each term separately. Here using the connection to the Z-transform will save us a lot:

$$\mathcal{Z}\left\{\left(\frac{1}{2}\right)^n \cos\left(\frac{\pi}{8}(n-1)\right) u[n]\right\} = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n \frac{1}{2}\left(e^{-j\pi/8}e^{j\pi n/8} + e^{j\pi/8}e^{-j\pi n/8}\right) z^{-n} \tag{65}$$

$$= e^{-j\pi/8}\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}e^{j\pi 8}z^{-1}\right)^n + e^{j\pi/8}\frac{1}{2}\left(\frac{1}{2}e^{-j\pi 8}z^{-1}\right)^n \tag{66}$$

$$= \frac{(1/2)e^{-j\pi/8}}{1 - (1/2)e^{j\pi/8}z^{-1}} + \frac{(1/2)e^{j\pi/8}}{1 - (1/2)e^{-j\pi/8}z^{-1}} \tag{67}$$

For the other term:

$$\mathcal{Z}\left\{\left(\frac{1}{2}\right)^{-n}\cos\left(\frac{\pi}{8}(n-1)\right)u[-n-1]\right\} = \sum_{n=-\infty}^{1}\left(\frac{1}{2}\right)^{|n|}\frac{1}{2}\left(e^{-j\pi/8}e^{j\pi n/8}+e^{j\pi/8}e^{-j\pi n/8}\right)z^{-n} \tag{68}$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n}\frac{1}{2}\left(e^{-j\pi/8}e^{-j\pi n/8}+e^{j\pi/8}e^{j\pi n/8}\right)z^{n} \tag{69}$$

$$= \frac{1}{2}e^{-j\pi/8}\sum_{n=1}^{\infty}\left(\frac{1}{2}e^{-j\pi/8}z\right)^{n} + \frac{1}{2}e^{j\pi/8}\sum_{n=1}^{\infty}\left(\frac{1}{2}e^{j\pi/8}z\right)^{n} \tag{70}$$

$$= \frac{1}{2}e^{-j\pi/8}\frac{\frac{1}{2}e^{-j\pi/8}z}{1-\frac{1}{2}e^{-j\pi/8}z} + \frac{1}{2}e^{j\pi/8}\frac{\frac{1}{2}e^{j\pi/8}z}{1-\frac{1}{2}e^{j\pi/8}z} \tag{71}$$

$$= \frac{(1/4)e^{-j\pi/4}z}{1-(1/2)e^{-j\pi/8}z} + \frac{(1/4)e^{j\pi/4}z}{1-(1/2)e^{j\pi/8}z} \tag{72}$$

Now adding the two terms together and plugging in $z = e^{j\omega}$,

$$X(e^{j\omega}) = \frac{(1/2)e^{-j\pi/8}}{1-(1/2)e^{j\pi/8}e^{-j\omega}} + \frac{(1/2)e^{j\pi/8}}{1-(1/2)e^{-j\pi/8}e^{-j\omega}}$$
$$\frac{(1/4)e^{-j\pi/4}e^{j\omega}}{1-(1/2)e^{-j\pi/8}e^{j\omega}} + \frac{(1/4)e^{j\pi/4}e^{j\omega}}{1-(1/2)e^{j\pi/8}e^{j\omega}} \tag{73}$$

(c) For this one we have to apply the definition:

$$X(e^{j\omega}) = -3e^{j3\omega} - 2e^{j2\omega} - 1e^{j\omega} + 1e^{-j\omega} + 2e^{-2j\omega} + 3e^{-j3\omega} \tag{74}$$

$$= -6j\sin(3\omega) - 4j\sin(2\omega) - 2j\sin(\omega). \tag{75}$$

(d) This one we can use the table:

$$X(e^{j\omega}) = -\frac{\pi}{j}\delta(\omega+\pi/2) + \frac{\pi}{j}\delta(\omega+\pi/2) + \pi\delta(\omega+1) + \pi\delta(\omega-1). \tag{76}$$