Solutions for Homework 4

Problem 1. Obtain the inverse Laplace transform of each of the following functions, assuming a causal time-domain signal:

(a) $X_1(s) = 2 + \frac{4(s-3)}{s^2+9}$

(b) $X_2(s) = \frac{4}{s} + \frac{4s}{s^2+16}$

(c) $X_3(s) = \frac{(s+5)e^{-2s}}{(s+1)(s+3)}$

Solution:

(a) We can use three transforms from the table:

$$\delta(t) \overset{\mathcal{L}}{\longleftrightarrow} 1 \tag{1}$$

$$\frac{s}{s^2+9} \overset{\mathcal{L}}{\longleftrightarrow} \cos(3t)u(t) \tag{2}$$

$$\frac{3}{s^2+9} \overset{\mathcal{L}}{\longleftrightarrow} \sin(3t)u(t) \tag{3}$$

Then

$$x_1(t) = 2\delta(t) + 4(\cos(3t)u(t) - \sin(3t))u(t) \tag{4}$$

$$= 2\delta(t) + 4\sqrt{2}\cos(3t + \pi/4)u(t). \tag{5}$$

Or we can use a combination property (see SSTA):

$$\cos(3t + \pi/2)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s/\sqrt{2} - 3/\sqrt{2}}{s^2+9}. \tag{6}$$

Then we have

$$2\delta(t) \overset{\mathcal{L}}{\longleftrightarrow} 2 \tag{7}$$

$$4\sqrt{2}\cos(3t + \pi/4)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{4(s-3)}{s^2+9}. \tag{8}$$

Then

$$x_1(t) = 2\delta(t) + 4\sqrt{2}\cos(3t + \pi/4)u(t). \tag{9}$$

(b) Again, using the table,

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s} \tag{10}$$

$$\cos(4t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2+16} \tag{11}$$

So

$$x_2(t) = 4u(t) + 4\cos(4t)u(t). \tag{12}$$

(c) The effect of e^{-2s} is a delay by -2, so we focus on the rational transfer function instead, which has poles at -1 and -3. Partial fraction expansion yields

$$\left.\frac{s+5}{s+3}\right|_{s=-1} = 2 \tag{13}$$

$$\left.\frac{s+5}{s+1}\right|_{s=-3} = -1 \tag{14}$$

$$\tag{15}$$

so

$$X_3(s) = e^{-2s}\left(\frac{2}{s+1} - \frac{1}{s+3}\right) \tag{16}$$

and

$$x_3(t) = \left(2e^{-(t-2)} - e^{-3(t-2)}\right)u(t-2). \tag{17}$$

Problem 2 (LSS Midterm 2, Fall 2018 Short Answer)**.** (a) Consider a system with Laplace transform

$$H(s) = \frac{s+1}{s^2+s-20}. \tag{18}$$

Is this system lowpass or bandpass? Explain your reasoning.

Solution:

(a) The poles are at $-\frac{1}{2} \pm \frac{9}{2}j$ and there is a zero at -1. This is a bit tricky and the right way to do it is to plot the frequency response in MATLAB. It's flat for low frequencies and then rises a bit due to the poles and then rolls off. You can argue that it is lowpass because it's flat at low frequencies, or bandpass since it has a bump near the poles. Credit was awarded for both answers to this question.

Problem 3. How many signals have a Laplace transform that may be expressed as

$$X(s) = \frac{s-1}{(s+2)(s+3)(s^2+s+1)}. \tag{19}$$

Find the ROC for each of these transforms.

Solution:

First we find the poles and zeros. There is a zero at $s = 1$ and poles at $s = -2, -3, -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$. So there are 4 regions of convergence: $-\infty < \mathfrak{Re}\{s\} < -3$, $-3 < \mathfrak{Re}\{s\} < -2$, and $-2 < \mathfrak{Re}\{s\} < -\frac{1}{2}$, and $-\frac{1}{2} < \mathfrak{Re}\{s\} < \infty$.

Problem 4. This one is like solving a mystery! We are given five facts about a real signal $x(t)$ with Laplace transform $X(s)$:

1. $X(s)$ has exactly 2 poles.
2. $X(s)$ has no zeros in the finite s-plane.
3. $X(s)$ has a pole at $s = -1 + j$.
4. $e^{2t}x(t)$ is *not* absolutely integrable.
5. $X(0) = 8$.

With these pieces of evidence in hand, find $X(s)$ and specify its region of convergence.

Solution:

To solve the mystery we have to use the clues at hand. First off, we know we have to find all the poles and zeros of $X(s)$. The first two facts tell us that we need to find 2 poles and if there are any zeros, they are at ∞. One of the poles is at $s = -1 + j$ so we have

$$\frac{C}{(s - (-1 + j))(s - b_2)}. \tag{20}$$

So now we have to find C and b_2. But we are also told that $x(t)$ is real, which means poles come in conjugate pairs, so if $s = -1 + j$ is a pole, then so is $s = -1 - j$:

$$\frac{C}{(s - (-1 + j))(s - (-1 - j))}. \tag{21}$$

This means the ROC can either be $\mathfrak{Re}\{s\} < -1$ or $\mathfrak{Re}\{s\} > -1$.

We're also told that $e^{2t}x(t)$ is not absolutely integrable, which means the ROC of $e^{2t}x(t)$ does not contain the imaginary axis. Multiplying by e^{2t} in the time domain is an s-shift by 2, so we know that the ROC shifted to the right does not contain the imaginary axis. This means that the ROC is $\mathfrak{Re}\{s\} > -1$ since $\mathfrak{Re}\{s\} > -1 + 2$ does not contain the imaginary axis.

Looking at our last fact, we can plug things in:

$$X(0) = \frac{C}{(1 - j)(1 + j)} = \frac{C}{2} = 8 \tag{22}$$

So $C = 16$. So finally,

$$X(s) = \frac{16}{(s - (-1 + j))(s - (-1 - j))} = \frac{16}{s^2 + 2s + 2}. \tag{23}$$

with ROC $\mathfrak{Re}\{s\} > -1$.

Problem 5. This problem how LTI systems can be used to model all sorts of real phenomena. When we build measurement systems (i.e. sensors), we often cannot control the relationship between the physical thing we want to measure and the output of the sensor. But by using some known inputs (steps, sinusoids, etc.) we can try to characterize the system behavior to post-process the "raw" sensor reading into the signal we actually want to measure.

A pressure gauge that can be modeled as an LTI system has a time response to a unit step input given by $(1 - e^{-t} - te^{-t})u(t)$. For a certain input $x(t)$ the output is observed to be $(2 - 3e^{-t} + e^{-3t})u(t)$.

From this observed measurement, determine the true pressure input to the gauge as a function of time.

Solution:

This seems sort of tricky (as all word problems are) until you start writing down some block diagrams and equations. We have the step response of a system:

$$(1 - e^{-t} - te^{-t})u(t) \tag{24}$$

Taking a Laplace transform:

$$\begin{aligned}
\frac{1}{s} - \frac{1}{s+1} + \frac{d}{ds}\frac{1}{s+1} &= \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2} && (25)\\
&= \frac{s^2 + 2s + 1 - s(s+1) - s}{s(s+1)^2} && (26)\\
&= \frac{1}{s(s+1)^2} && (27)\\
&= \frac{1}{s}H(s), && (28)
\end{aligned}$$

where in the last line we used the convolution property to write the step response as $\frac{1}{s}$ times the impulse response of the system. This means $H(s) = \frac{1}{(s+1)^2}$.

Now we are told that

$$(x * h)(t) = (2 - 3e^{-t} + e^{-3t})u(t) \tag{29}$$

So

$$\begin{aligned}
X(s)H(s) &= \frac{2}{s} - \frac{3}{s+1} + \frac{1}{s+3} && (30)\\
&= \frac{2(s+1)(s+3) - 3s(s+3) + s(s+1)}{s(s+1)(s+3)} && (31)\\
&= \frac{2s^2 + 8s + 6 - 3s^2 - 9s + s^2 + s}{s(s+1)(s+3)} && (32)\\
&= \frac{6}{s(s+1)(s+3)} && (33)\\
X(s) &= \frac{6(s+1)}{s(s+3)}. && (34)
\end{aligned}$$

Doing partial fraction expansion,

$$\begin{aligned}
\left.\frac{6(s+1)}{(s+3)}\right|_{s=0} &= 2 && (35)\\
\left.\frac{6(s+1)}{s}\right|_{s=-3} &= 4 && (36)
\end{aligned}$$

So

$$x(t) = (2 + 4e^{-3t})u(t). \tag{37}$$

Problem 6 (Oppenheim, Willsky, and Nawab 10.21)**.** Determine the z-transform for each for the following sequences:

(a) $\left(-\frac{1}{3}\right)^n u[-n-2]$

(b) $2^n u[-n] + \left(\frac{1}{4}\right)^n u[n-1]$

(c) $\left(\frac{1}{2}\right)^{n+1} u[n+3]$

(d) $\left(\frac{1}{3}\right)^{n-2} u[n-2]$

Solution:

(a) We can apply the definition and so some series manipulations:

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} && (38)\\
&= \sum_{n=-\infty} -2(-1/3)^n z^{-n} && (39)\\
&= \sum_{n=2}^{\infty}(-1/3)^{-n} z^n && (40)\\
&= \sum_{n=0}^{\infty}(-1/3)^{-n-2} z^{n+2} && (41)\\
&= \frac{9z^2}{1+3z}, \qquad |z| < 1/3 && (42)\\
&= \frac{3z}{1+\frac{1}{3}z^{-1}}, \qquad |z| < 1/3. && (43)
\end{aligned}$$

(b) We first want to split this up into two terms. For the first,

$$\begin{aligned}
\mathcal{Z}\left\{2^n u[-n]\right\} &= \sum_{n=-\infty}^{0} 2^n z^{-n} && (44)\\
&= \sum_{n=0}^{\infty} 2^{-n} z^n && (45)\\
&= \frac{1}{1-\frac{1}{2}z}, \qquad |z| < 2 && (46)\\
&= \frac{-2z^{-1}}{1-2z^{-1}}, \qquad |z| < 2. && (47)
\end{aligned}$$

For the second term:

$$\begin{aligned}
\mathcal{Z}\left\{\left(\frac{1}{4}\right)^n u[n-1]\right\} &= \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n z^{-n} && (48)\\
&= \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{n+1} z^{-n-1} && (49)\\
&= \frac{\frac{1}{4}z^{-1}}{1-\frac{1}{4}z^{-1}}, \qquad |z| > \frac{1}{4} && (50)
\end{aligned}$$

So the sum has the ROC that is the intersection of the ROCs of the two terms:

$$X(z) = \frac{-2z^{-1}}{1-2z^{-1}} + \frac{\frac{1}{4}z^{-1}}{1-\frac{1}{4}z^{-1}}, \qquad \frac{1}{4} < |z| < 2 \tag{51}$$

(c) This is a straightforward application of the definition:

$$X(z) = \sum_{n=-3}^{\infty} \left(\frac{1}{2}\right)^{n+1} z^{-n} \tag{52}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n-2} z^{-n+3} \tag{53}$$

$$= \frac{4z^3}{1 - \frac{1}{2}z^{-1}}, \qquad |z| > \frac{1}{2}. \tag{54}$$

(d) We can use similar tricks:

$$X(z) = \sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^{n-2} z^{-n} \tag{55}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n} z^{-n-2} \tag{56}$$

$$= \frac{z^{-2}}{1 - \frac{1}{3}z^{-1}}, \qquad |z| > \frac{1}{3}. \tag{57}$$

Problem 7. Find the z-transforms of the following signals (including the ROC).

(a) A chunk of a decaying exponential:

$$x[n] = \begin{cases} a^n, & 0 \leq n \leq N-1, a > 0 \\ 0, & \text{otherwise} \end{cases} \tag{58}$$

(b) A two-sided exponential:

$$y[n] = b^{|n|}, \qquad b > 0. \tag{59}$$

Solution:

(a) We can just use the definition to write this as a partial power series:

$$X(z) = 1 + az^{-1} + a^2 z^{-2} + \cdots + a^{N-1} z^{N-1} \tag{60}$$

$$= \sum_{k=1}^{N-1} (az^{-1})^k \tag{61}$$

If $|z| > a$ we can simplify:

$$= \frac{1 - a^N z^{-N}}{1 - az^{-1}}. \tag{62}$$

However, in general we have to leave the formula in terms of (61). Since this is a finite-length signal the ROC is the whole plane (except 0).

(b) For this one we can write things out in terms of two sums:

$$Y(z) = \cdots + b^n z^n + \cdots + bz + 1 + bz^{-1} + b^2 z^{-2} + \cdots + b^n z^{-n} + \cdots \tag{63}$$

$$= -1 + \sum_{k=0}^{\infty} (bz)^n + \sum_{k=0}^{\infty} (bz^{-1})^n \tag{64}$$

For both of these infinite series to converge we need $|z| < \frac{1}{b}$ and $|z| > b$. This will only work if $b < 1$, otherwise the two inequalities are incompatible and one series must diverge.

Problem 8. Consider the signal

$$x[n] = \begin{cases} \left(\frac{1}{3}\right)^n \cos\left(\frac{\pi}{4}n\right), & n \leq 0 \\ 0, & n > 0 \end{cases} \tag{65}$$

Determine the poles and ROC for $X(z)$.

Solution:

There are several ways to do this. You can look it up from a table, but that raises the question: how did they get the result in the table? So instead let's try to do is using Eulerization and applying the basic transform for exponential signals. Let's make things more general first, and set $r = \frac{1}{3}$ and $\omega_0 = \pi/4$ so we have

$$\begin{aligned} x[n] &= r^n \cos(\omega_0 n)\, u[n] && (66)\\ &= \frac{1}{2} r^n e^{j\omega_0 n} u[n] + \frac{1}{2} r^n e^{-j\omega_0 n} u[n] && (67)\\ &= \frac{1}{2}\left(re^{j\omega_0}\right)^n u[n] + \frac{1}{2}\left(re^{-j\omega_0}\right)^n u[n] && (68) \end{aligned}$$

So now $X(z)$ can be found from the transform pair $a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}}$:

$$\begin{aligned} X(z) &= \frac{1/2}{1 - re^{j\omega_0 n} z^{-1}} + \frac{1/2}{1 - re^{-j\omega_0 n} z^{-1}} && (69)\\ &= \frac{1}{2}\left(\frac{1 - re^{-j\omega_0} z^{-1} + 1 - re^{j\omega_0} z^{-1}}{(1 - re^{j\omega_0} z^{-1})(1 - re^{-j\omega_0} z^{-1})}\right) && (70)\\ &= \frac{1}{2}\left(\frac{2 - 2r\cos(\omega_0) z^{-1}}{(1 - r(e^{j\omega_0} + e^{-j\omega_0}) z^{-1} + r^2 z^{-2})}\right) && (71)\\ &= \frac{1 - r\cos(\omega_0) z^{-1}}{1 - 2r\cos(\omega_0) z^{-1} + r^2 z^{-2}} && (72) \end{aligned}$$

This has a zero at $z = r\cos(\omega_0)$ and poles at

$$\begin{aligned} z^{-1} &= \frac{2r\cos(\omega_0) \pm \sqrt{4r^2\cos^2(\omega_0) - 4r^2}}{2r^2} && (73)\\ &= \frac{1}{r}\cos(\omega_0) \pm \frac{1}{r}\sqrt{\cos^2(\omega_0) - 1} && (74)\\ &= \frac{1}{r}\cos(\omega_0) \pm \frac{1}{r} j\sin(\omega_0) && (75)\\ &= \frac{1}{r} e^{\pm j\omega_0} && (76) \end{aligned}$$

Or $z = re^{j\omega_0}$ and $z = re^{-j\omega_0}$. The ROC for this problem is $|z| > r$ to make the power series converge earlier.

Plugging in our values, we get poles at $\frac{1}{3}e^{j\pi/4}$ and $\frac{1}{3}e^{-j\pi/4}$ and a zero at $\frac{1}{3}\cos(\pi/4)$ with an ROC of $|z| > \frac{1}{3}$.