Solutions for Homework 3

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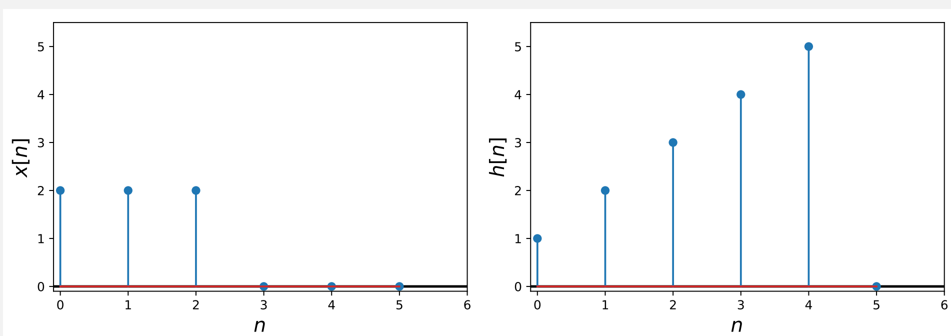
Problem 1. Find the convolution of the following two DT sequences:

$$x[k] = \begin{cases} 2 & 0 \le k \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$h[k] = \begin{cases} k+1 & 0 \le k \le 4 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Solution:

This problem is a bit of a test to see if you're willing to *draw a picture*:



Looking at this graphically, we can see the output is just the superposition of three copies of $2h[k]$ delayed by 0, 1, and 2, or 2 times sum of three consecutive values of $h[k]$:

$$\begin{aligned} y[k] &= 2h[k] + 2h[k-1] + 2h[k-2] && (3) \\ &= 2\delta[k] + 6\delta[k-1] + 12\delta[k-2] + 18\delta[k-3] + 24\delta[k-4] + 18\delta[k-5] + 10\delta[k-6]. && (4) \end{aligned}$$

Again, the key steps are to draw a picture and use the interpretation of the convolution instead of just blindly applying the formula: convolving a signal h with a constant signal just adds up consecutive values h and linearity gives you the multiplication by 2.

Problem 2 (ECE 345 Fall 2017, Midterm 1)**.** For LTI systems the system is BIBO stable if its impulse response is *absolutely summable*:

$$\sum_{n=-\infty}^{\infty} |h_i[n]| < \infty \qquad \Longrightarrow \qquad \text{BIBO-stable} \tag{5}$$

Suppose we have a DT LTI system defined by the following input-output relation:

$$y[n] = x[n] + \frac{1}{3}x[n-1]. \tag{6}$$

Calculate the *inverse system*. Is it causal? Is it stable? *Hint:* see if you can write $x[n]$ in terms of $y[n]$, $y[n-1]$, etc. and try to find a pattern.

Solution:

This uses a recursive trick:

$$x[n] = y[n] - \frac{1}{3}x[n-1] \tag{7}$$

$$x[n-1] = y[n-1] - \frac{1}{3}x[n-2] \tag{8}$$

$$x[n] = y[n] - \frac{1}{3}y[n-1] + \frac{1}{3^2}x[n-2] \tag{9}$$

$$x[n-2] = y[n-2] - \frac{1}{3}x[n-3] \tag{10}$$

$$x[n] = y[n] - \frac{1}{3}y[n-1] + \frac{1}{3^2}y[n-2] - \frac{1}{3^3}x[n-3] \tag{11}$$

$$x[n] = \sum_{k=0}^{\infty} \left(-\frac{1}{3}\right)^k y[n-k]. \tag{12}$$

The inverse system is thus

$$h_i[n] = \sum_{k=0}^{\infty} \left(-\frac{1}{3}\right)^k \delta[n-k]. \tag{13}$$

This is absolutely summable:

$$\sum_{n=0}^{\infty} |h_i[n]| = \sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1-1/3} = \frac{3}{2} < \infty. \tag{14}$$

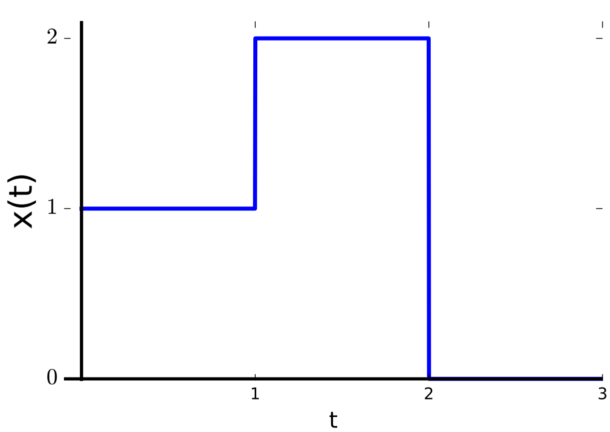
so the inverse system is stable. The impulse response is only $\neq 0$ for $n \geq 0$ so it is also causal.

Problem 3 (*ECE 345 Midterm 1, Fall 2018*)**.** Suppose the impulse response of an LTI system is given by

$$h(t) = e^{-2t}(u(t-1) - u(t-3)). \tag{15}$$

(a) Sketch the impulse response of the signal. Be sure to label all important points.

(b) Suppose we put the following input signal $x(t)$ into the channel.



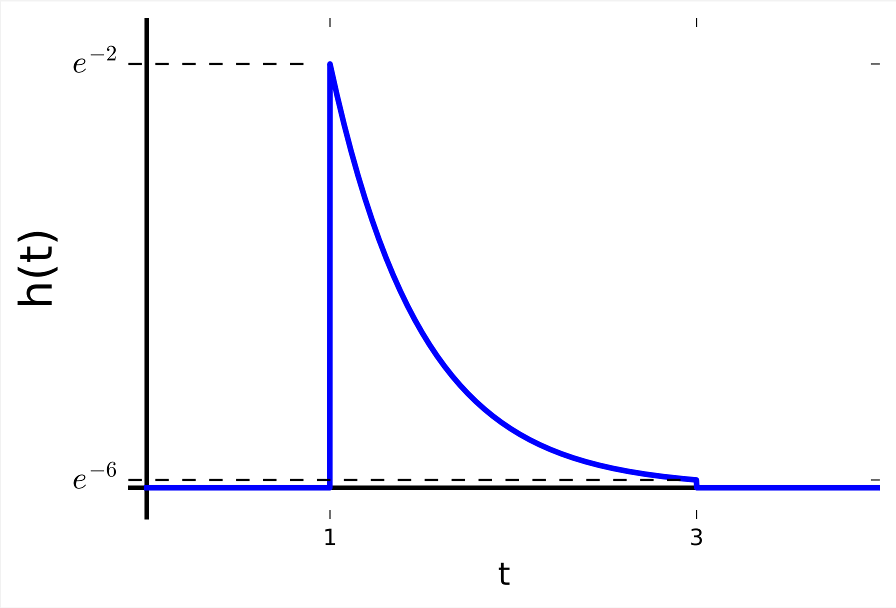
Write an algebraic expression for $x(t)$ in terms of unit step functions.

(c) For the $x(t)$ in the previous part, compute the output of the LTI system $h(t)$ with input $x(t)$. *Hint:* use linearity.

(d) Suppose we instead apply the input $w(t) = e^{-t}$ into this system $h(t)$. Calculate the output of the system.

Solution:

(a) Here is a plot:



(b) The function is

$$x(t) = u(t) + u(t-1) - 2u(t-2). \tag{16}$$

(c) We only need to find the convolution $h(t) * u(t)$ – the rest follows from LTI system properties. The output is 0 until time $t = 1$. From $t = 1$ to $t = 3$ the convolution integral yields

$$\int_1^t e^{-2\tau} d\tau = -\frac{1}{2}e^{-2\tau}\Big|_{t=1}^{t} = \frac{1}{2}e^{-2} - \frac{1}{2}e^{-2t} \tag{17}$$

Then at time $t = 3$ the integral is constant:

$$\int_1^3 e^{-2\tau} d\tau = \frac{1}{2}e^{-2} - \frac{1}{2}e^{-6} \tag{18}$$

Let $w(t) = (h * u)(t)$. Now to use linearity and time invariance, the output is

$$y(t) = w(t) + w(t-1) - 2w(t-2) \tag{19}$$

$$= \begin{cases} 0 & t < 1 \\ \frac{1}{2}e^{-2} - \frac{1}{2}e^{-2t} & 1 \leq t < 2 \\ \frac{1}{2}e^{-2} - \frac{1}{2}e^{-2t} + \frac{1}{2}e^{-2} - \frac{1}{2}e^{-2(t-1)} & 2 \leq t < 3 \\ \frac{1}{2}e^{-2} - \frac{1}{2}e^{-6} + \frac{1}{2}e^{-2} - \frac{1}{2}e^{-2(t-1)} - e^{-2} - e^{-2(t-2)} & 3 \leq t < 4 \\ e^{-2} - e^{-6} - e^{-2} - e^{-2(t-2)} & 4 \leq t < 5 \\ 0 & t > 5 \end{cases} \tag{20}$$

Simplifying:

$$y(t) = \begin{cases} 0 & t < 1 \\ \frac{1}{2}e^{-2} - \frac{1}{2}e^{-2t} & 1 \leq t < 2 \\ e^{-2} - \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-2(t-1)} & 2 \leq t < 3 \\ -\frac{1}{2}e^{-6} - \frac{1}{2}e^{-2(t-1)} - e^{-2(t-2)} & 3 \leq t < 4 \\ -e^{-6} - e^{-2(t-2)} & 4 \leq t < 5 \\ 0 & t > 5 \end{cases} \tag{21}$$

(d) Since exponentials are eigenfunctions of LTI systems, we need the Laplace transform evaluated at $s = -1$. The Laplace transform of $h(t)$ is

$$H(s) = \int_1^3 e^{at}e^{-st}dt = \frac{1}{s-a}e^{(-s+a)t}\Big|_1^3 \tag{22}$$

$$= \frac{1}{s-a}(e^{-3(s-a)} - e^{-(s-a)}) \tag{23}$$

Plugging in $a = -2$ for our $h(t)$ we get

$$H(s) = \frac{1}{s+2}(e^{-3(s+2)} - e^{-(s+2)}). \tag{24}$$

Now setting $s = -1$:

$$H(-1) = (e^{-3} - e^{-1}) \tag{25}$$

$$\tag{26}$$

So the output is

$$y(t) = (e^{-3} - e^{-1})e^{-t}. \tag{27}$$

Problem 4. Suppose a CT LTI system has impulse response

$$h(t) = (3e^{-2t} - 2e^{-4t})u(t) \tag{28}$$

Compute the output of the system with the following inputs:

(a) $x(t) = e^{-3t}u(t)$

(b) $x(t) = 2e^{-2t}u(t)$

Solution:

One way to solve this problem is to solve a simpler problem first and then build up the answer from that simpler problem. You can then keep the simpler problem's solution on hand as a reference to just look up (e.g. on a cheat sheet for an exam).

Looking at this problem you should first note that since we're dealing with LTI systems we can write out the convolution as:

$$(h * x)(t) = 3e^{-2t}u(t) * x(t) - 2e^{-4t}u(t) * x(t). \tag{29}$$

Since both of the $x(t)$s in this problem are also of the form ce^{-at} so we can compute

$$(e^{-at}u(t)) * (e^{-bt}u(t)) \tag{30}$$

and build up the answer from there. For $a \neq b$:

$$\begin{aligned}(e^{-at}u(t)) * (e^{-bt}u(t)) &= \int_{-\infty}^{\infty} e^{-a\tau}e^{-b(t-\tau)}u(\tau)u(t-\tau)d\tau && (31)\\ &= \int_0^t e^{-bt}e^{-(a-b)\tau}d\tau && (32)\\ &= \frac{1}{b-a}e^{-bt}e^{-(a-b)\tau}\Big|_0^t && (33)\\ &= \frac{1}{b-a}e^{-at} + \frac{1}{a-b}e^{-bt}. && (34)\end{aligned}$$

For $a = b$:

$$\begin{aligned}(e^{-at}u(t)) * (e^{-at}u(t)) &= \int_{-\infty}^{\infty} e^{-a\tau}e^{-a(t-\tau)}u(\tau)u(t-\tau)d\tau && (35)\\ &= \int_0^t e^{-at}d\tau && (36)\\ &= te^{-at}. && (37)\end{aligned}$$

As a sanity check, because convolution is commutative, if we switch a and b we should get the same answer. Now, armed with this, we can just plug into the formula.

(a) We have

$$\begin{aligned}e^{-2t}u(t) * e^{-3t}u(t) &= e^{-2t} - e^{-3t} && (38)\\ e^{-4t}u(t) * e^{-3t}u(t) &= -e^{-4t} + e^{-3t} && (39)\\ (3e^{-2t} - 2e^{-4t})u(t) * e^{-3t}u(t) &= 3(e^{-2t} - e^{-3t}) - 2(-e^{-4t} + e^{-3t}) && (40)\\ &= 2e^{-4t} - 5e^{-3t} + 3e^{-2t}. && (41)\end{aligned}$$

(b) We have

$$\begin{aligned}e^{-2t}u(t) * e^{-2t}u(t) &= te^{-2t} && (42)\\ e^{-4t}u(t) * e^{-2t}u(t) &= \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-4t} && (43)\\ (3e^{-2t} - 2e^{-4t})u(t) * 2e^{-2t}u(t) &= 6te^{-2t}u(t) - 4\left(\frac{1}{2}e^{-2t}u(t) - \frac{1}{2}e^{-4t}u(t)\right) && (44)\\ &= 6te^{-2t}u(t) - 2e^{-2t}u(t) + 2e^{-4t}u(t) && (45)\end{aligned}$$

Problem 5 (SSTA 2.15)**.** It's important to be able to compute convolutions without having to resort to the definition each time. The key is to use convolutions which you have computed before. Here are some to practice on. Remember: draw a picture!

(a) $u(t) * [2u(t) - 2u(t-3)]$

(b) $u(t) * [(t-1)u(t-1)]$

(c) $[\delta(t) + 2\delta(t-1) + 3\delta(t-2)] * [4\delta(t) + 5\delta(t-1)]$

Solution:

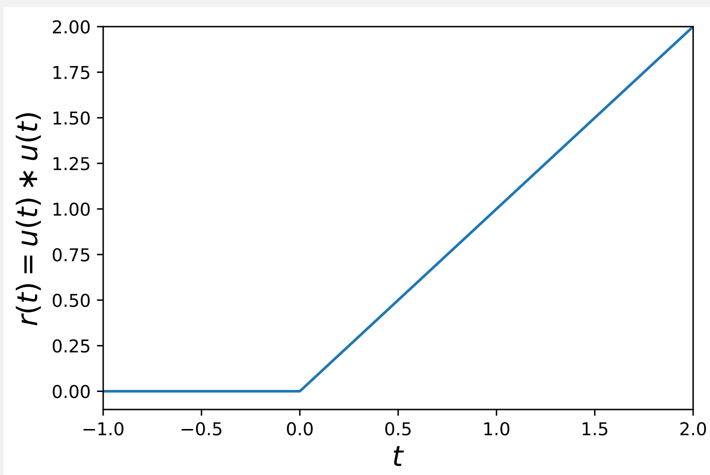
The key facts we need to solve this are that the ramp function $r(t) = u(t) * u(t)$:

Instead of just trying to remember this as a fact to memorize, you should think of how to interpret convolution with a unit step function: what does convolution with $u(t)$ mean? Looking at the formula,

$$(x * u)(t) = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} x(\tau)d\tau, \tag{46}$$

so the convolving $x(t)$ with $u(t)$ produces the *integral* of $x(t)$: this is the thing you should remember.

What's the convolution of $u(t)$ with itself? Well, you just integrate the unit step, which gives you the unit ramp. Here's a plot:



(47)

What's the convolution of $tu(t)$ with $u(t)$? Note that $tu(t) = r(t)$ so this is just $\int t dt = \frac{1}{2}t^2 u(t)$.

For the last one you need to use the property that convolving with $\delta(t-\tau)$ produces a "copy" of the signal delayed by τ. The rest of the problem is using the linearity and time invariance properties.

(a) By linearity:

$$u(t) * [2u(t) - 2u(t-3)] = 2(u(t) * u(t)) - 2(u(t) * u(t-3)) \tag{48}$$

By time-invariance, the first term produces an $r(t)$ and the second an $r(t-3)$:

$$u(t) * [2u(t) - 2u(t-3)] = 2r(t) - 2r(t-3). \tag{49}$$

(b) By time-invariance:

$$u(t) * [(t-1)u(t-1)] = (u * r)(t-1) \tag{50}$$

$$= \frac{1}{2}(t-1)^2 u(t-1). \tag{51}$$

(c) Looking at the two terms it seems easier to make delayed copies using the second one:

$$[\delta(t) + 2\delta(t-1) + 3\delta(t-2)] * [4\delta(t) + 5\delta(t-1)] = 4(\delta(t) + 2\delta(t-1) + 3\delta(t-2)) + 5(\delta(t-1) + 2\delta(t-2) + 3\delta(t-3)) \tag{52}$$

$$= 4\delta(t) + 13\delta(t-1) + 22\delta(t-2) + 15\delta(t-3). \tag{53}$$

Problem 6. Consider the signal

$$x(t) = e^{-5t}u(t)e^{-\beta t}u(t), \tag{54}$$

and denote its Laplace transform by $X(s)$. What are the constraints placed on the real and imaginary parts of β if the region of convergence of $X(s)$ is $\mathfrak{Re}(s) > -3$?

Solution:

If we take the Laplace transform of this signal, we get

$$X(s) = \mathcal{L}\left\{e^{-(5+\beta)t}u(t)\right\} \tag{55}$$

$$= \frac{1}{s + (5+\beta)}. \tag{56}$$

So to get $\mathfrak{Re}\{s\} > -3$ we need $\mathfrak{Re}\{\beta\} = -2$. However, since the ROC is made up of vertical strips, the imaginary part $\mathfrak{Im}\{\beta\}$ is unconstrained.

Problem 7 (SSTA 3.8)**.** Determine the Laplace transform of the following functions:

(a) $x_1(t) = \frac{d}{dt}\left(4te^{-2t}\cos(4\pi t + \pi/6)u(t)\right)$

(b) $x_2(t) = e^{-3t}\cos(4t + \pi/6)u(t)$

(c) $x_3(t) = t^2(u(t) - u(t-4))$

(d) $x_4(t) = 10\cos(6\pi t + \pi/6)\delta(t - 0.2)$

Solution:

First we need to use the Laplace transform of a phase shifted cosine:

$$\mathcal{L}\left\{\cos(\omega_0 t + \theta)\right\} = \frac{s\cos(\theta) - \omega_0\sin(\theta)}{s^2 + \omega^2}. \tag{57}$$

(a) We can define

$$x_a(t) = 4\cos(4\pi t + \pi/6)u(t) \tag{58}$$

$$x_b(t) = e^{-2t}u(t) \tag{59}$$

$$x_c(t) = tx_b(t) \tag{60}$$

$$x_1(t) = \frac{d}{dt}x_c(t) \tag{61}$$

$$X_a(s) = 4\frac{s\cos(\pi/6) - 4\pi\sin(\pi/6)}{s^2 + (4\pi)^2} \tag{62}$$

$$= \frac{2\sqrt{3}s - 8\pi}{s^2 + (4\pi)^2} \tag{63}$$

Now define

$$X_b(s) = X_a(s+2) \tag{64}$$
$$= \frac{2\sqrt{3}(s+2) - 8\pi}{(s+2)^2 + (4\pi)^2} \tag{65}$$
$$X_c(s) = -\frac{d}{ds}X_b(s) \tag{66}$$
$$= -\frac{2\sqrt{3}((s+2)^2 + (4\pi)^2) - (2\sqrt{3}(s+2) - 8\pi)2(s+2)}{((s+2)^2 + (4\pi)^2)^2} \tag{67}$$
$$= \frac{2\sqrt{3}(s+2)^2 - 16\pi(s+2) + 2\sqrt{3}(4\pi)^2}{((s+2)^2 + (4\pi)^2)^2} \tag{68}$$

Finally,

$$X_1(s) = sX_c(s) - x_c(0^-) \tag{69}$$
$$= s\frac{2\sqrt{3}(s+2)^2 - 16\pi(s+2) + 2\sqrt{3}(4\pi)^2}{((s+2)^2 + (4\pi)^2)^2} \tag{70}$$

You could simplify further and then take roots to find the pole-zero diagram etc. but that would be too messy by hand and you'd be better off using a computer at this point.

(b) Define

$$x_a(t) = \cos(4t + \pi/6)u(t) \tag{71}$$
$$= \cos(4t + \pi/6)u(t). \tag{72}$$

So

$$X_a(s) = \frac{s\cos(\pi/6) - 4\sin(\pi/6)}{s^2 + 16}. \tag{73}$$

Now apply the s-shift property:

$$X_2(s) = X_a(s+3) = \frac{(\sqrt{3}/2)(s+3) - 2}{(s+3)^2 + 16}. \tag{74}$$

(c) We can write $x_3(t)$ in two terms:

$$x_3(t) = t^2u(t) - t^2u(t-4). \tag{75}$$

Now we can use a different trick to deal with the second term – it's not in the form $f(t-T)u(t-T)$ but we can add and subtract $(8t - 18)$:

$$x_3(t) = t^2u(t) - (t-4)^2u(t-4) - 8(t-4)u(t-4) - 18u(t-4). \tag{76}$$

Now apply the Laplace transform to each term.

$$X_3(s) = \frac{2}{s^3} - \frac{2e^{-4s}}{s^3} - \frac{8e^{-4s}}{s^2} - \frac{16e^{-4s}}{s}. \tag{77}$$

(d) Start with the definition and use the sampling/sifting property of delta functions:

$$X_4(s) = \int_{0^-}^{\infty} 10\cos(6\pi t + \pi/6)\delta(t-0.2)e^{-st}dt \tag{78}$$
$$= 10\cos(6\pi t + \pi/6)e^{-st}\big|_{t=0.2} \tag{79}$$
$$= 10\cos(1.2\pi + \pi/6)e^{-0.2s} \tag{80}$$
$$= -4.1e^{-0.2s}. \tag{81}$$

Problem 8 (ECE 345 Fall 2019 Midterm 2)**.** Suppose a causal CT LTI system has bilateral Laplace transform

$$H(s) = \frac{2s-2}{s^2 + (10/3)s + 1}. \tag{82}$$

(a) Write the linear constant coefficient differential equation (LCCDE) relating a general input $x(t)$ to its corresponding output $y(t)$ of the system corresponding to this transfer function in equation (82).

(b) Plot the pole-zero diagram and indicate the region of convergence for this system. Is this system stable? Explain your answer.

(c) Suppose the input $x(t) = e^{-t}u(t)$. Find the output $y(t)$.

Solution:

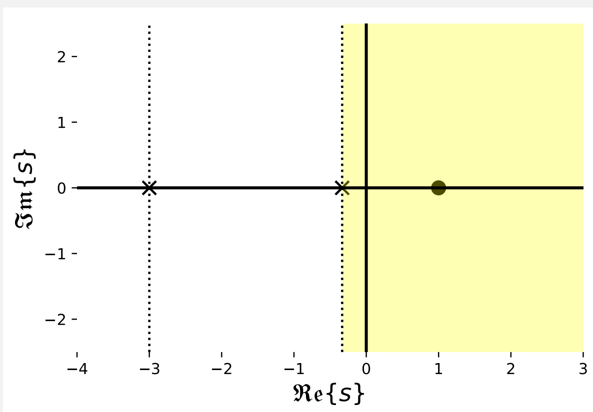
(a) Since $H(s) = \frac{Y(s)}{X(s)}$ we just cross-multiply and convert multiplication by s to differentiation in the time domain:

$$(s^2 + (10/3)s + 1)Y(s) = (2s-2)X(s) \tag{83}$$
$$\frac{d^2}{dt^2}y(t) + \frac{10}{3}\frac{d}{dt}y(t) + y(t) = 2\frac{d}{dt}x(t) - 2x(t) \tag{84}$$

(b) Factorizing:

$$H(s) = \frac{2s-2}{s^2 + \frac{10}{3}s + 1} = \frac{(2s-2)}{(s+\frac{1}{3})(s+3)}, \tag{85}$$

so there are poles at $-1/3$ and -3 and a zero at $+1$. Since the system is causal (given in the problem statement) the ROC is $\mathfrak{Re}\{s\} > -1/3$. Here is the pole-zero diagram:



(c) We take the Laplace transform $\frac{1}{s+1}$ of the input signal and multiply, then do partial fraction expansion:

$$Y(s) = \frac{(2s-2)}{(s+\frac{1}{3})(s+1)(s+3)} \tag{86}$$

$$c_1 = \frac{(2(-1/3)-2)}{(-1/3+1)(-1/3+3)} = -3/2 \tag{87}$$

$$c_2 = \frac{(2(-1)-2)}{(-1+\frac{1}{3})(-1+3)} = 3 \tag{88}$$

$$c_3 = \frac{(2(-3)-2)}{(-3+\frac{1}{3})(-3+1)} = -3/2 \tag{89}$$

$$Y(s) = \frac{-3/2}{(s+\frac{1}{3})} + \frac{3}{(s+1)} + \frac{-3/2}{(s+3)} \tag{90}$$

Taking the causal inverse of each term (since the output is causal because the input and system are both causal):

$$y(t) = -\frac{3}{2}e^{-(1/3)t}u(t) + 3e^{-t}u(t) - \frac{3}{2}e^{-3t}u(t). \tag{91}$$