Linear Systems and Signals

Feedback for CT systems

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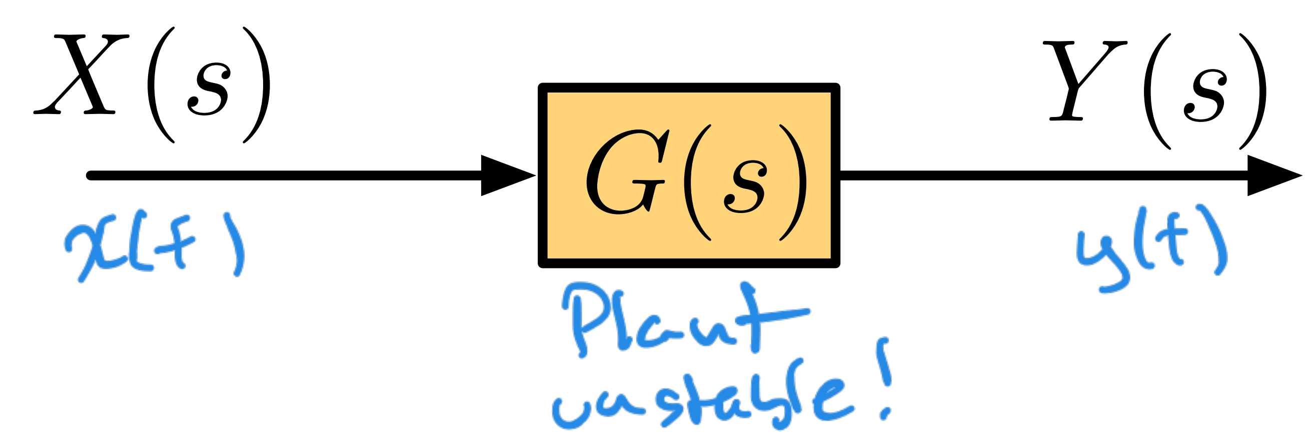
Learning objectives

The learning objectives for this section are:

- find the transfer function for a system with feedback
- use different control polices to create new transfer functions

Unstable systems



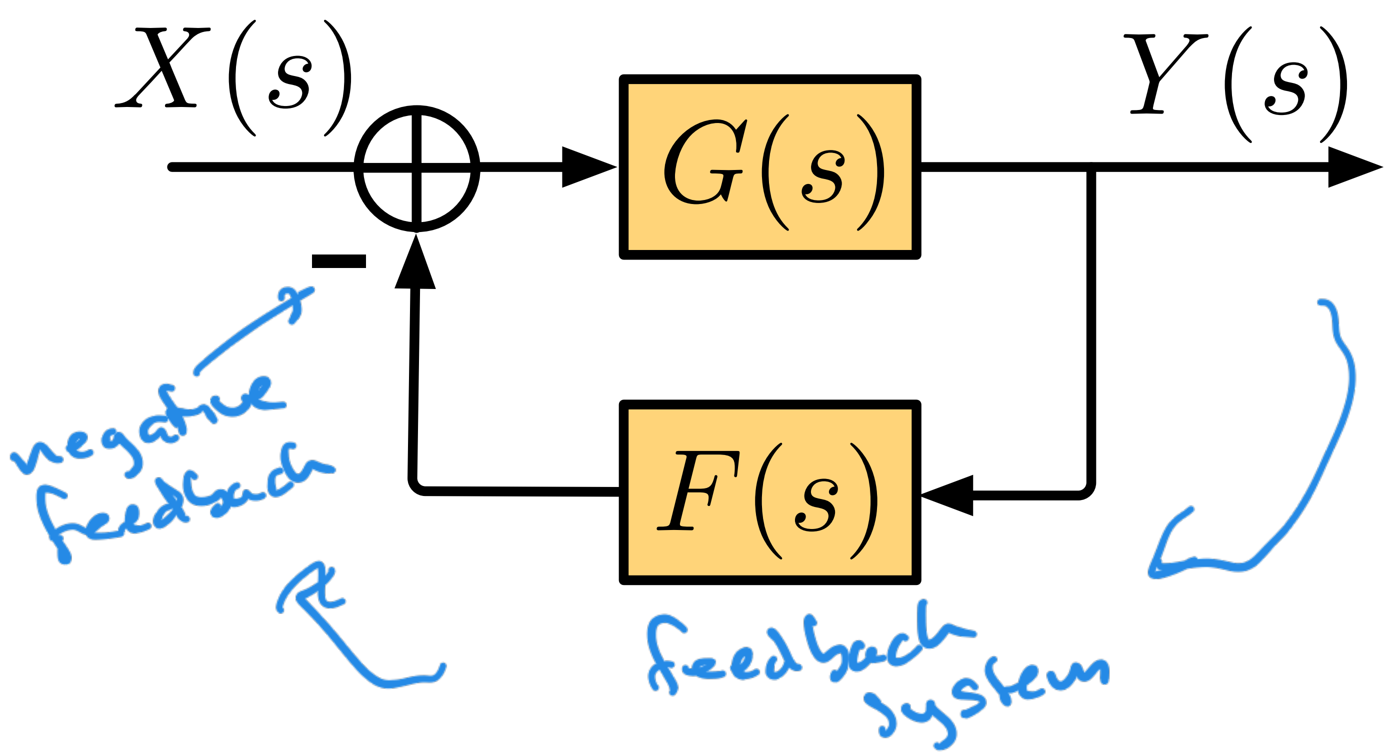
If the ROC does not contain the imaginary axis then the system is unstable. For causal systems, this means we have a pole in the right half-plane. A simple example:

$$H(s) = \frac{1}{s-a} \quad (1)$$

pole @ $s=a$ $a>0$

Engineering requires dealing with systems that are "naturally" unstable: we call this original system a *plant* (this term comes from chemical engineering). Control systems are studied in many other types of engineering, especially mechanical and industrial engineering.

The plant: an open loop system

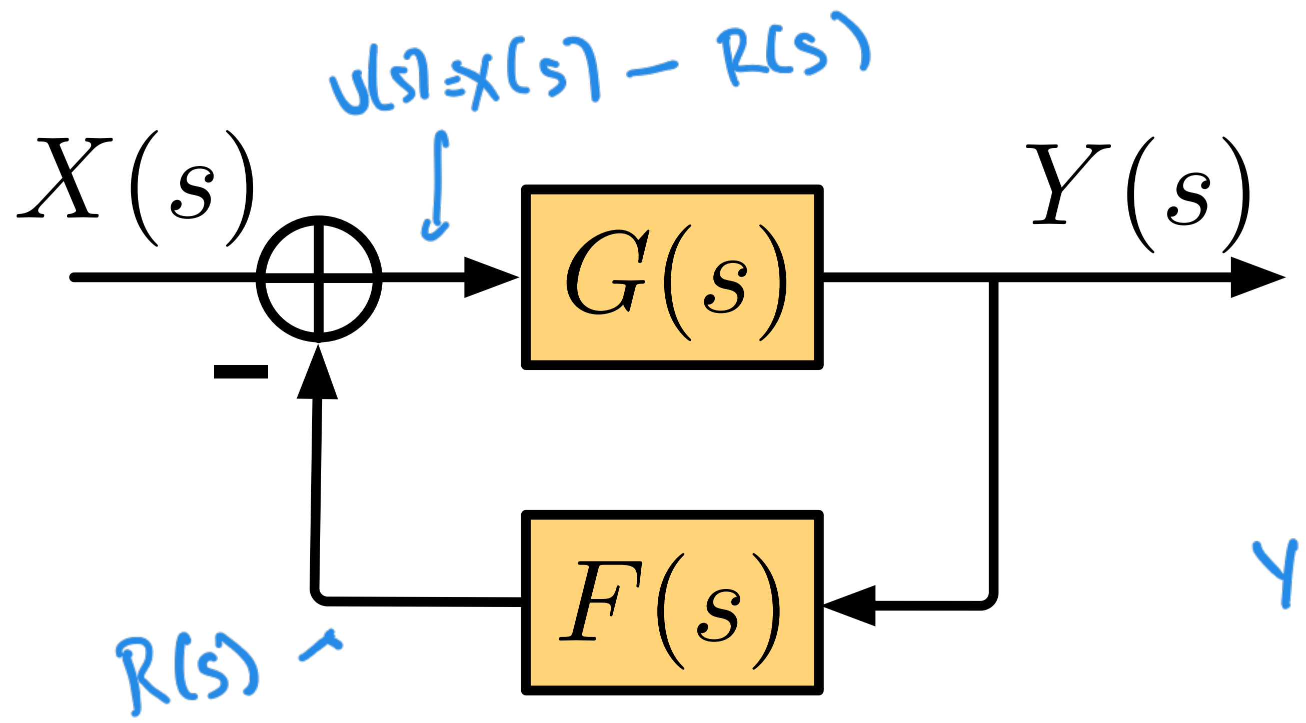


The plant is what we call an *open-loop* system. We would like to use *feedback* to "close the loop" and stabilize the system.

- use the output of the open loop system $G(s)$
- process it with an LTI feedback system $F(s)$
- subtract the result from the input (using *negative feedback*)

We have to find the overall transfer function for the closed loop system.

Example



R(s) = F(s) Y(s)
Y(s) = G(s) V(s)
V(s) = X(s) − R(s)

The output of $F(s)$ is $F(s)Y(s)$ and the input to $G(s)$ is $X(s) - F(s)Y(s)$. So we have

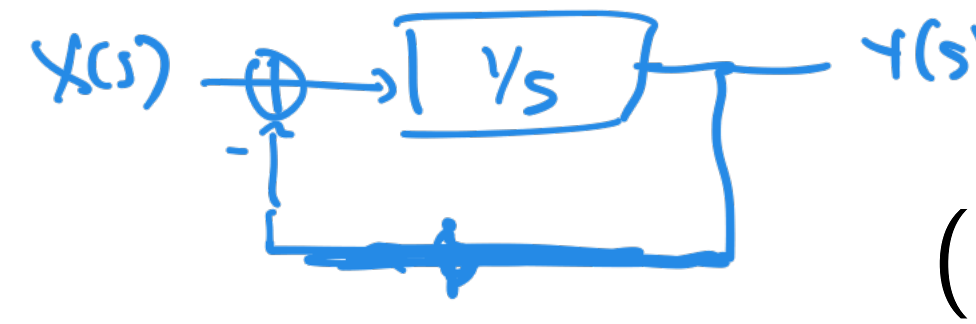
$$Y(s) = G(s)(X(s) - F(s)Y(s)) \tag{2}$$

$$(1 + F(s)G(s))Y(s) = G(s)X(s) \tag{3}$$

$$\boxed{H(s) = \frac{Y(s)}{X(s)} = \frac{G(s)}{1 + F(s)G(s)}} \tag{4}$$

This shows how feedback can possibly help: the poles of $H(s)$ will in general be different than those of $G(s)$.

Black's formula



$$H(s) = \frac{G(s)}{1 + F(s)G(s)} \tag{5}$$

This relation is known as *Black's formula* and shows how negative feedback control changes the transfer function $G(s)$ of the open-loop system to the transfer function $H(s) = \frac{G(s)}{1+F(s)G(s)}$ of the closed-loop system.

unstable

As an example, take $G(s) = \frac{1}{s}$ (an integrator) and $F(s) = 1$, which means we just connect the output of the system back to the input of the system. Then

$$H(s) = \frac{1/s}{1 + 1/s} = \frac{1}{s+1} \tag{6}$$

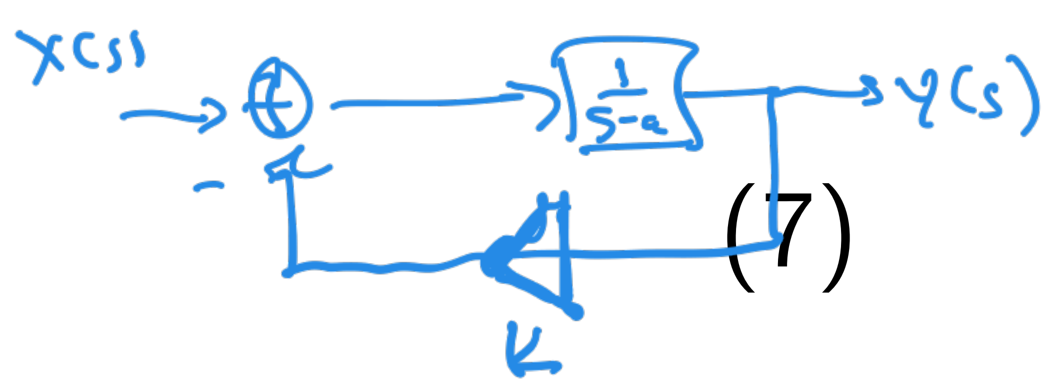
stable!
pole @ $s = -1$

The closed-loop system is stable!

Single pole: simple feedback

pole @ $s=a$ $a>0$

Does the same trick work with $G(s) = \frac{1}{s-a}$? Taking $F(s) = 1$:



$$H(s) = \frac{\overbrace{\frac{1}{s-a}}^{G(s)}}{1 + \underbrace{\frac{1}{s-a}}_{G(s)F(s)}} = \frac{1}{s-a+1}. \qquad (7)$$

This is only stable if $a < 1$. If instead we let $F(s) = K$

$$H(s) = \frac{\overbrace{\frac{1}{s-a}}^{G(s)}}{1 + \underbrace{\frac{K}{s-a}}_{G(s)F(s)}} = \boxed{\frac{1}{s-a+K}} = \frac{1}{s-a+K} \qquad (8)$$

This is stable if we choose $K > a$. The case $F(s) = K$ is called *proportional control* since the feedback signal is proportional to the output signal.

Example of proportional control

Let's try proportional control $F(s) = K$ on the plant

unstable!

$$G(s) = \frac{1}{s^2 + 3s - 10} = \frac{1}{(s+5)(s-2)}. \tag{9}$$

pole @ -5 pole @ +2

Then

$$H(s) = \frac{G(s)}{1 + F(s)G(s)} = \frac{\frac{1}{s^2+3s-10}}{1 + \frac{K}{s^2+3s-10}} \tag{10}$$

$$\boxed{= \frac{1}{s^2 + 3s - 10 + K}} \tag{11}$$

req. a gain of 10

So if $K > 10$ we can make the system stable since the poles will be in the right half-plane (hint: look at the quadratic formula).

Proportional-integral-derivative (PID) control

There are three forms of control that people often use for feedback systems:

- Proportional (P) control $F(s) = K_p$ as we have seen before.
- Derivative (D) control $F(s) = K_d s$ where we use the derivative of the output $y(t)$.
- Integral (I) control $F(s) = K_i \frac{1}{s}$ where we use the integral of the output $y(t)$.

In general we can use all three (PID control) but sometimes using two is sufficient (PI control or PD control). Implementation involves being able to build circuits/systems which can do integration or differentiation.

Example of PD control

Let's look at the same plant as before and look at PD control $F(s) = K_p + K_d s$:

$$G(s) = \frac{1}{s^2 + 3s - 10} \tag{12}$$

$$H(s) = \frac{\frac{1}{s^2+3s-10}}{1 + \frac{K_p + K_d s}{s^2+3s-10}} \tag{13}$$

$$= \frac{1}{s^2 + (3 + K_d)s + (K_p - 10)} \tag{14}$$

Here we still need $K_p > 10$ but the poles of the stabilized system have moved left (can you see why?) so the resulting system is more stable.

poles are farther left w/ $K_d > 0$

Try it yourself

Problem

Try using proportional (P) and proportional-derivative (PD) control to stabilize the following open-loop systems.

- $G(s) = \frac{3}{s(s+4)}$ *(this is a good model for a motor)*
- $G(s) = \frac{1}{(s-2)(s-3)(s+8)}$ *(this has three poles)*
- $G(s) = \frac{1}{s^3+2s+1}$ *(show this cannot be stabilized with PD)*