Linear Systems and Signals
The ROC for Laplace transforms

Anand D. Sarwate

Department of Electrical and Computer Engineering
Rutgers, The State University of New Jersey

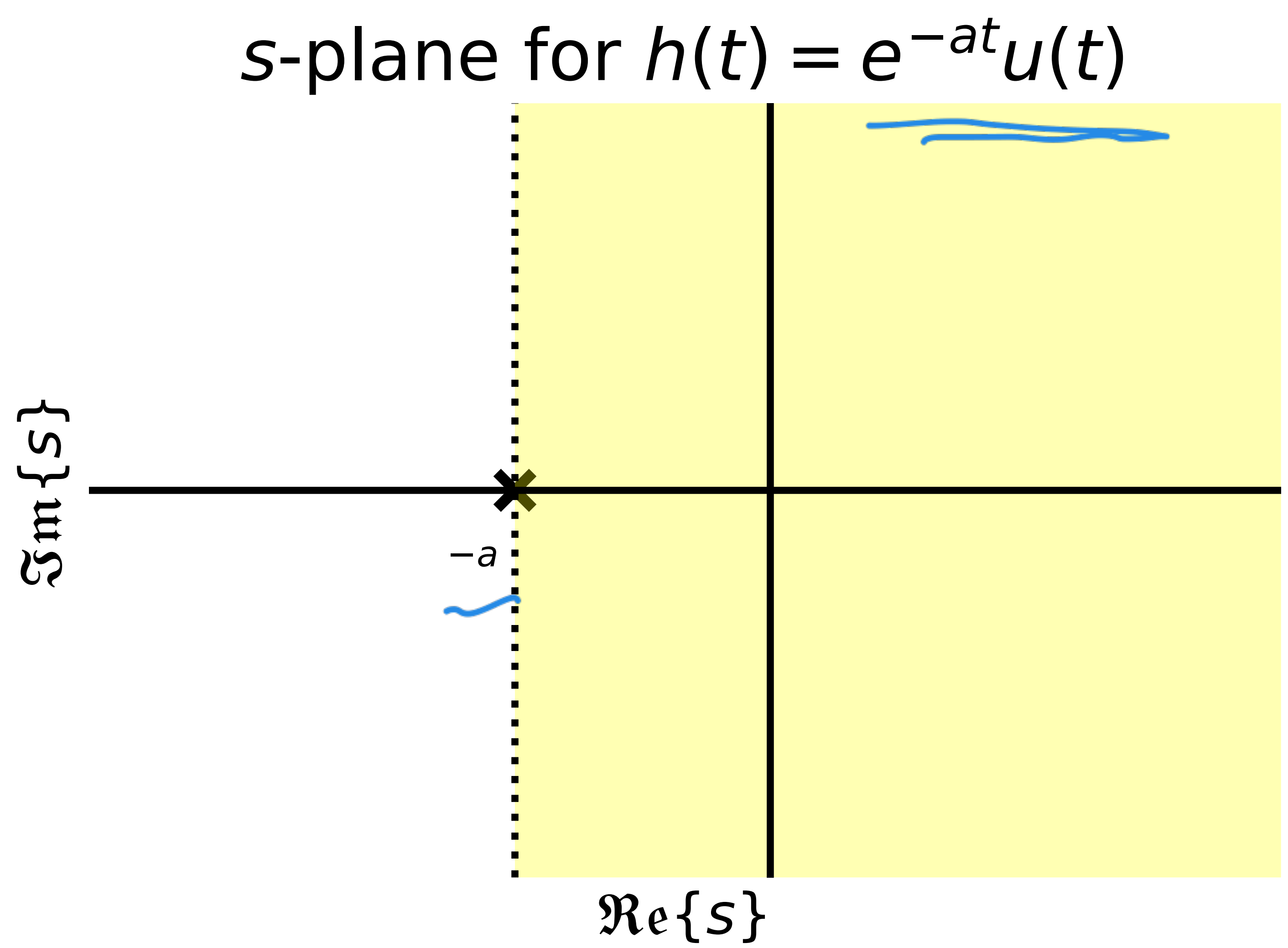
2020

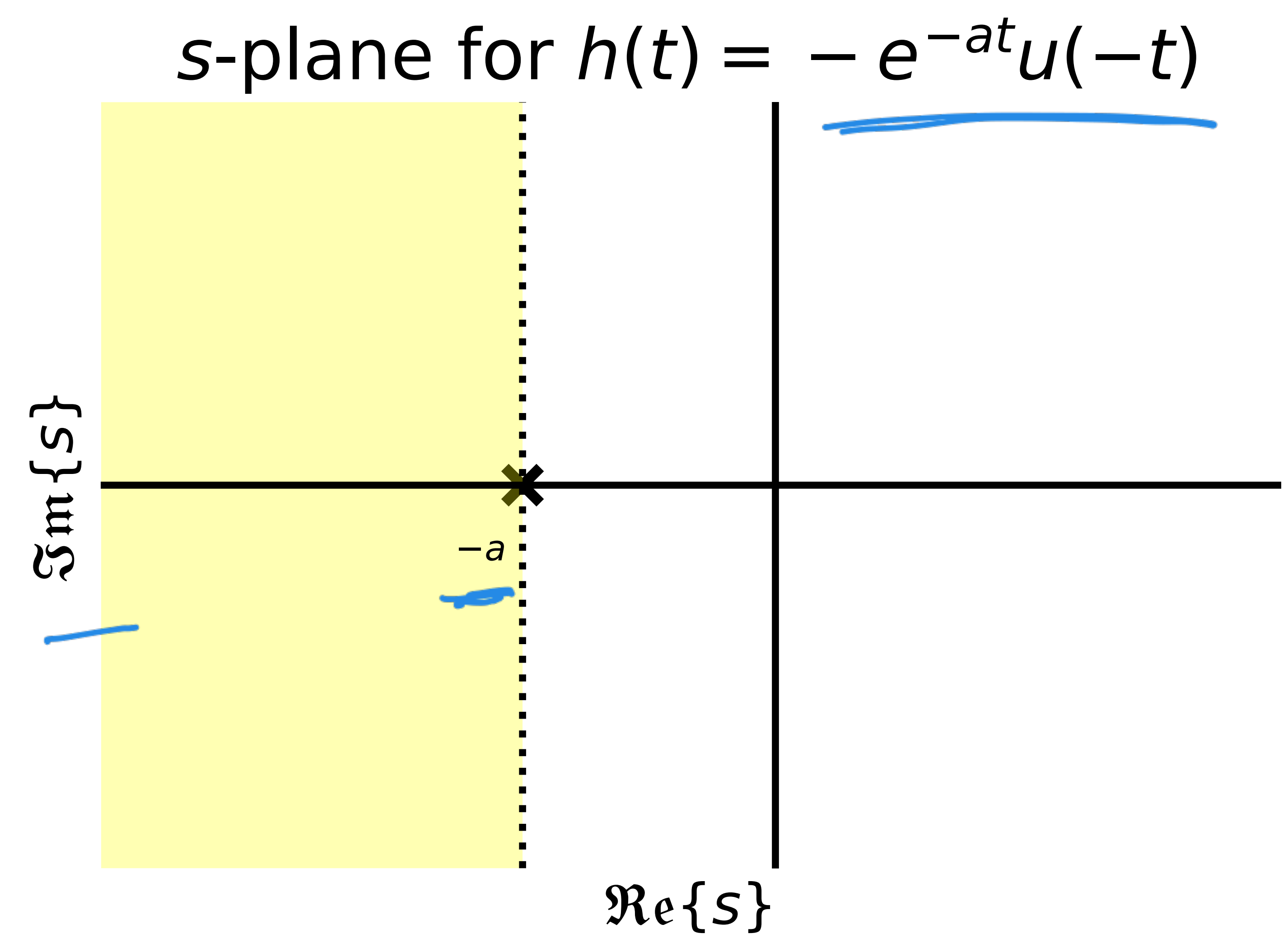
Learning objectives

The learning objectives for this section are:

- identify the region of convergence for a Laplace transform
- apply properties of the ROC for Laplace transforms

The Region of Convergence





We saw earlier

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} \qquad -e^{-at}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} \tag{1}$$

A Laplace transform needs the formula and the ROC.

Poles and zeros: rational transforms

A *pole-zero diagram* is a picture of s-plane marked with two types of points:

- $\times$ marks a *pole*, where $H(s)$ blows up to $\pm\infty$
- $\circ$ marks a *zero*, where $H(s) = 0$.

The ROC cannot contain any poles since the Laplace transform integral diverges at poles.

in s

A *rational* Laplace transform is the ratio of polynomials:

$$H(s) = \frac{a_k s^k + a_{k-1} s^{k-1} + \cdots + a_1 s + a_0}{b_\ell s^\ell + b_{\ell-1} s^{\ell-1} + \cdots + b_1 s + b_0} \quad (2)$$

$$= C \frac{(s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_k)}{(s-\beta_1)(s-\beta_2)\cdots(s-\beta_\ell)} \quad (3)$$

This has poles at $\{\beta_i\}$ and zeros at $\{\alpha_j\}$.

Linear combinations of complex exponentials (LCCEs)

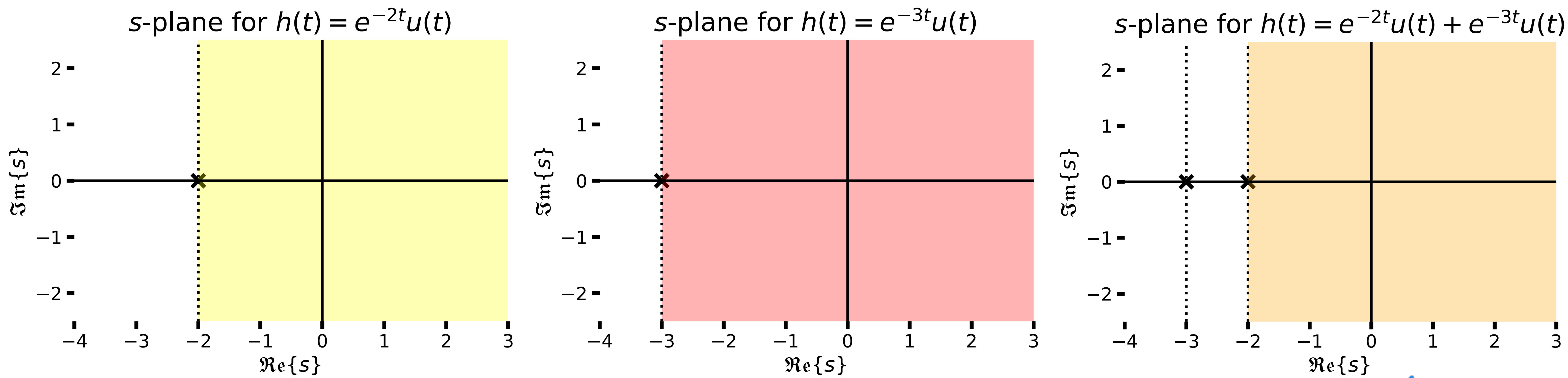
A major theme in this class is representing a signal as a *linear combination of complex exponentials* (LCCE). The signals $\cos(\omega t)$ and $\sin(\omega t)$ are LCCEs (think of Eulerizing). Rational Laplace transforms come from taking the Laplace transform of LCCEs. This is because the Laplace transform is *linear*:

$$\int_{-\infty}^{\infty} \left(\sum_{j=1}^{\ell} c_j e^{\beta_j t} \right) e^{-st} dt = \sum_{j=1}^{\ell} c_j \int_{-\infty}^{\infty} e^{\beta_j t} e^{-st} \tag{4}$$

$$= \sum_{j=1}^{\ell} \frac{c_j}{s - \beta_j} \tag{5}$$

But what about the ROC?

ROC for sums of exponentials



Let's look at something simpler:

both

$$h(t) = e^{-2t}u(t) + e^{-3t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2} + \frac{1}{s+3} \qquad (6)$$

We need both terms to converge so we need $\mathfrak{Re}\{s\} > -2$ and $\mathfrak{Re}\{s\} > -3$. Since one ROC is contained in the other we get the intersection, which is just $\mathfrak{Re}\{s\} > -2$.

ROC for sums

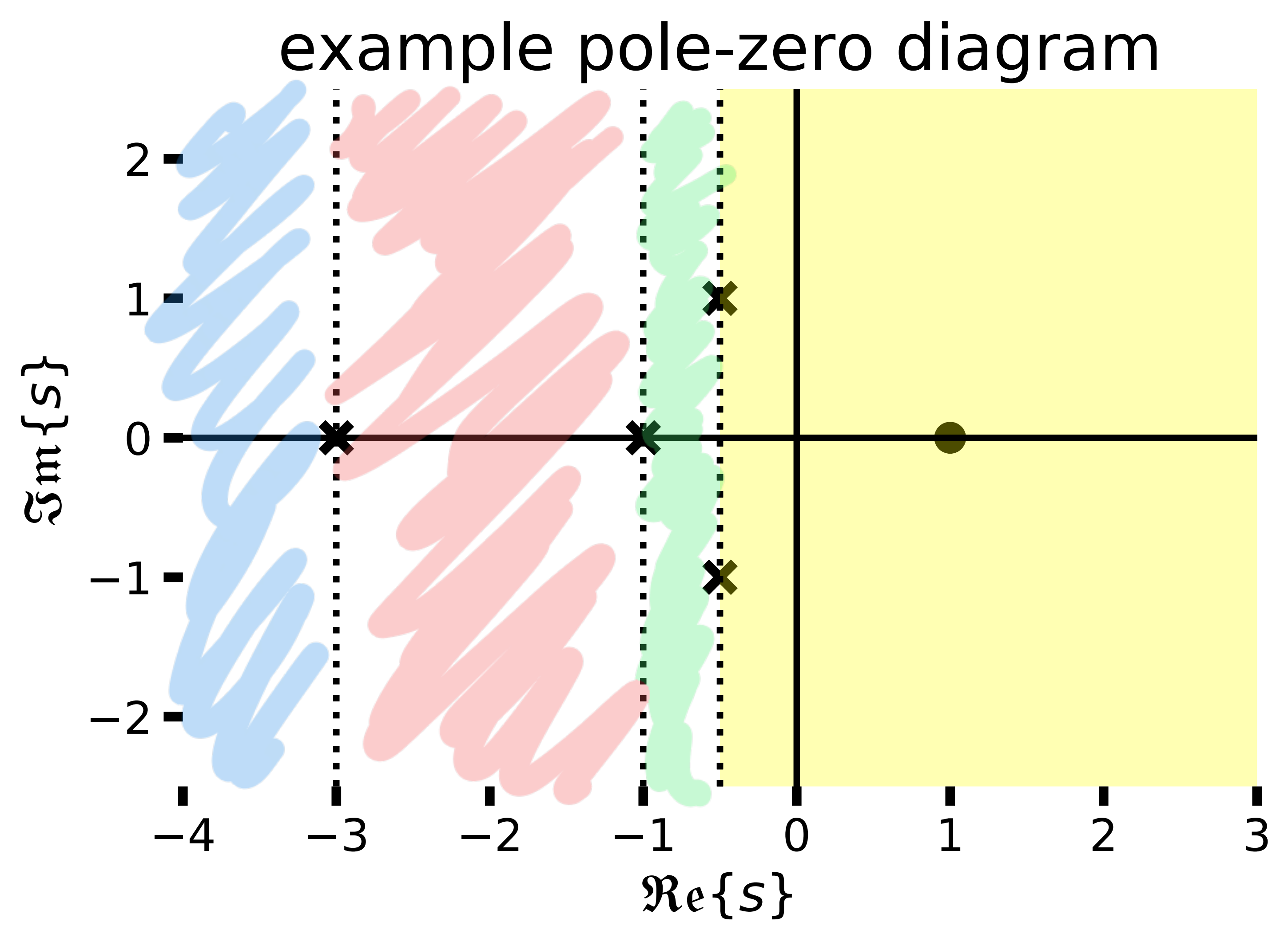
In general, if $\mathcal{R}_1$ is the ROC of $H_1(s)$ and $\mathcal{R}_2$ is the ROC for $H_2(s)$ then the ROC of $H_1(s) + H_2(s)$ *contains* the intersection $\mathcal{R}_1 \cap \mathcal{R}_2$.

The reason we say contains and not equals is that in the sum $H_1(s) + H_2(s)$ we could get some *pole-zero cancellation* – one of the terms in the combined denominator can get factored out.

- This is in a sense a mathematical trick – pole-zero cancellation requires exactly matching the pole and zero locations.
- For practical systems, it is nearly impossible to implement a pole-zero cancellation.

To see an example, try $\boxed{\frac{1}{s+1} + \frac{1}{(s+1)(s+2)}}$.

Rules for ROCs

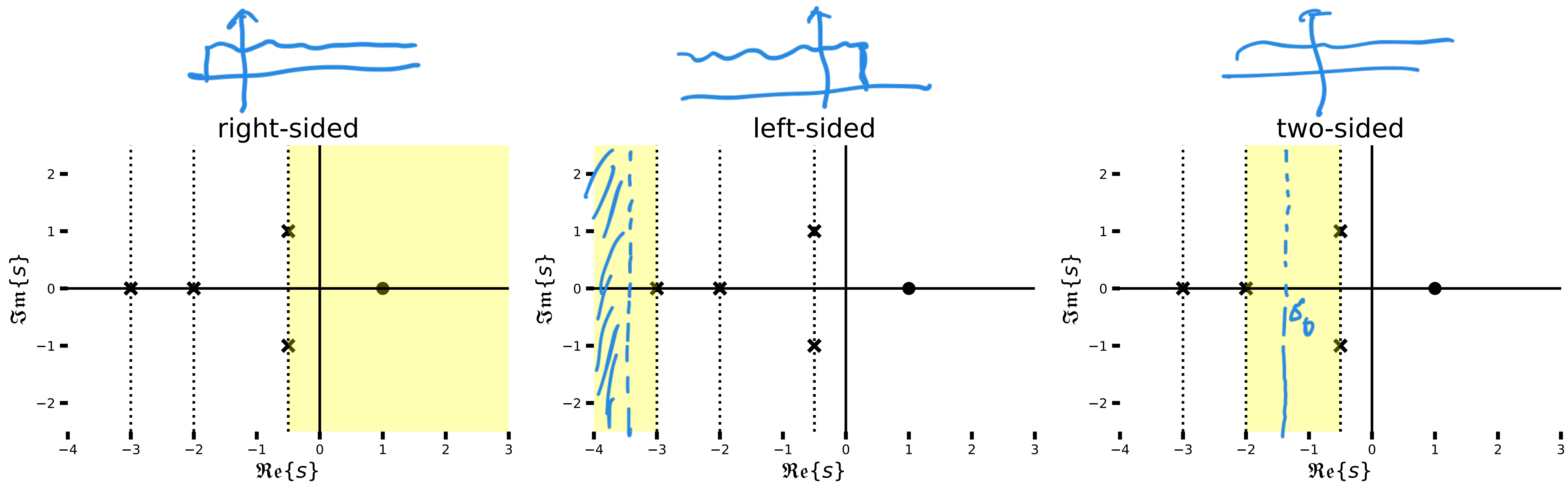


Here are some basic rules for any ROC $\mathcal{R}$:

1. $\mathcal{R}$ is always made of vertical strips parallel to the imaginary axis.
2. $\mathcal{R}$ cannot contain any poles.
3. If $h(t)$ is finite duration and $\int_{-\infty}^{\infty} |h(t)| dt < \infty$ (absolutely integrable), then $\mathcal{R} = \mathbb{C}$.

system is stable

ROCs: sidedness



We call a signal *right-sided* if $h(t) = 0$ for $t < T_0$ and *left-sided* if $h(t) = 0$ for $t > T_0$.

If the vertical line $\{s : \Re e\{s\} = \sigma_0\}$ is in $\mathcal{R}$ then

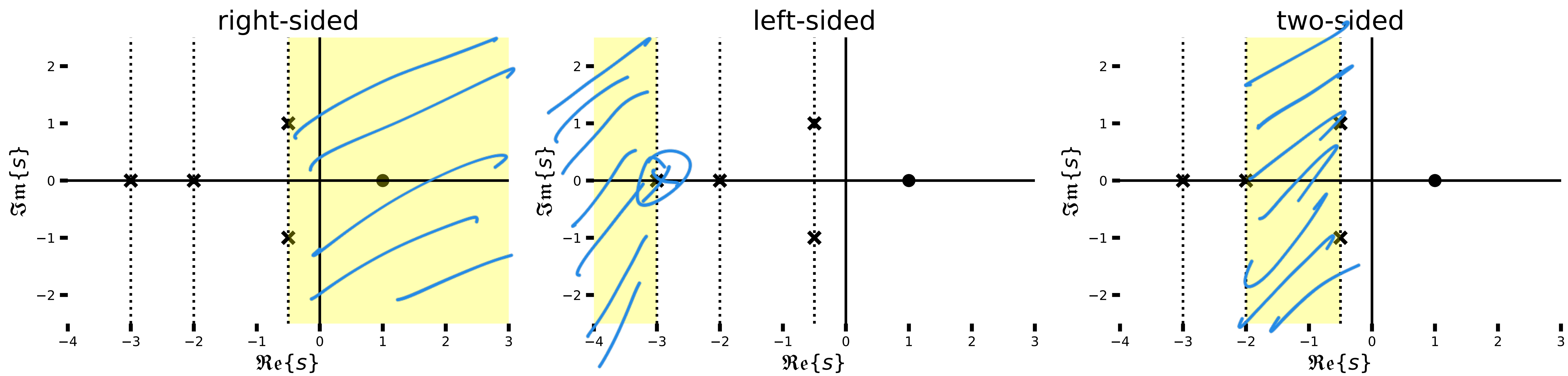
plane to the right of σ_0

1. if $h(t)$ is right-sided then $\{s : \Re e\{s\} > \sigma_0\} \subset \mathcal{R}$
2. if $h(t)$ is left-sided then $\{s : \Re e\{s\} < \sigma_0\} \subset \mathcal{R}$

plane to the left of σ_0

3. if $h(t)$ is two-sided then $\mathcal{R}$ is a strip containing $\{s : \Re e\{s\} = \sigma_0\}$.

Rational transforms and pole-zero diagrams



If $H(s)$ is rational then $\mathcal{R}$ is bounded by poles or extends to $+\infty$ or $-\infty$:

- If $h(t)$ is right-sided then $\mathcal{R}$ is to the right of the right-most pole.
- If $h(t)$ is left-sided then $\mathcal{R}$ is to the left of the ~~right~~-most pole.

left

Try it yourself

Problem

Find the Laplace transform, ROC, and pole-zero diagrams for the following signals

$$h(t) = 2e^{-4t}u(t) - 3e^{-3t}u(t) \tag{7}$$

$$x(t) = \cos(4\pi t)u(t) + 2e^{-5t} \tag{8}$$

$$r(t) = e^{-t}\sin(6\pi t) - e^{-2t}\cos(3\pi t) \tag{9}$$