Linear Systems and Signals

Finding system outputs for rational Laplace transforms

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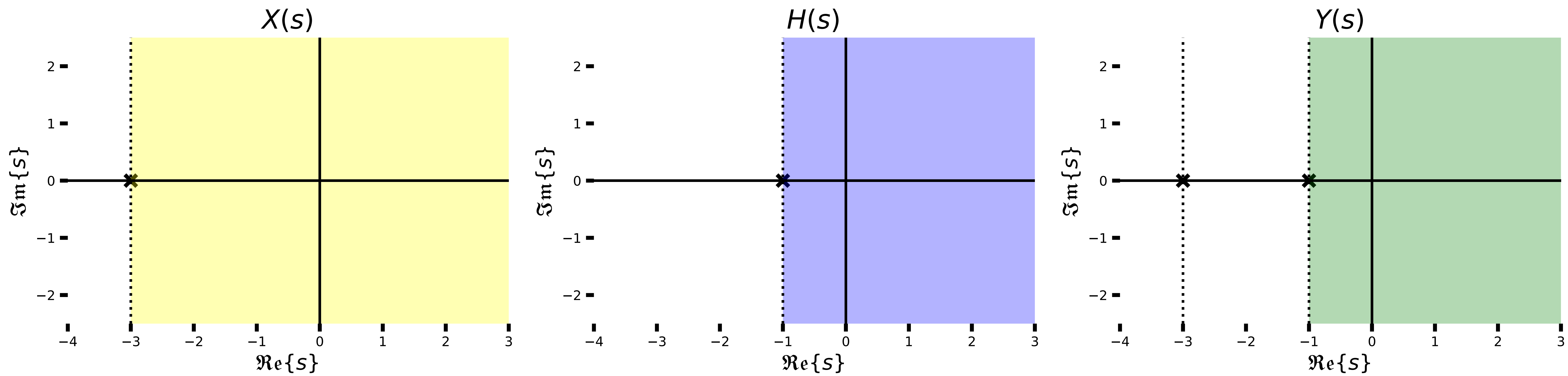
Learning objectives

The learning objectives for this section are:

- find the Laplace transform of a system output given an input and impulse response
- use partial fraction expansion to get the time-domain output signal

System outputs for rational transforms



Recall that

$$y(t) = (h * x)(t) \overset{\mathcal{L}}{\longleftrightarrow} H(s)X(s) = Y(s) \qquad (1)$$

Suppose

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \qquad \mathcal{R}_x \qquad (2)$$

$$h(t) \overset{\mathcal{L}}{\longleftrightarrow} H(s) \qquad \mathcal{R}_h \qquad (3)$$

If $H(s)X(s) = Y(s)$ has no pole-zero cancellations, $\mathcal{R}_y = \mathcal{R}_x \cap \mathcal{R}_h$.

Example

Suppose

$\frac{1}{s+2}$

$$x(t) = e^{-2t}u(t) \tag{4}$$

$$h(t) = -e^{-t}u(t) + e^{-3t}u(t) \tag{5}$$

We want to find $y(t)$ and $Y(s)$ and its ROC.
First step: take the Laplace transforms of $x(t)$ and $h(t)$. Using the transform table:

$\mathfrak{Re}\{s\} > -1$ $\mathfrak{Re}\{s\} > -3$

$$X(s) = \frac{1}{s+2} \qquad \mathcal{R}_h\{\mathfrak{Re}\{s\} > -2\} \tag{6}$$

$$H(s) = \frac{-1}{s+1} + \frac{1}{s+3} \qquad \boxed{\mathcal{R}_h = \{\mathfrak{Re}\{s\} > -1\}} \tag{7}$$

$$= \frac{-2}{(s+1)(s+3)} \qquad \{\mathfrak{Re}\{s\} > -1\} \tag{8}$$

$$\tag{9}$$

Finding the output transform

So the output is

$$Y(s) = H(s)X(s) = \frac{-2}{(s+1)(s+2)(s+3)} \qquad \underline{\{\mathfrak{Re}\{s\} > -1\}} \tag{10}$$

$$\tag{11}$$

To get the time-domain signal we use *partial fraction expansion*:

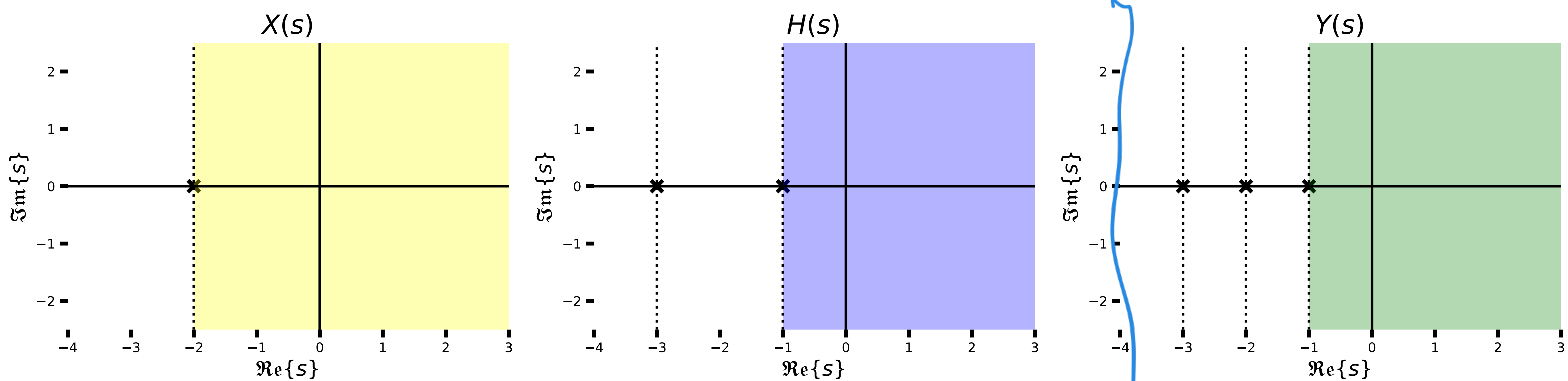
$$\left.\frac{-2}{(s+2)(s+3)}\right|_{s=-1} = -1 \tag{12}$$

$$\left.\frac{-2}{(s+1)(s+3)}\right|_{s=-2} = 2 \tag{13}$$

$$\left.\frac{-2}{(s+1)(s+2)}\right|_{s=-3} = -1 \tag{14}$$

$$\tag{15}$$

Converting to the time domain



The ROC is $\{\mathfrak{Re}\{s\} > -1\}$ and

$$\boxed{Y(s) = \frac{-1}{s+1} + \frac{2}{s+2} - \frac{1}{s+3}} \tag{16}$$

so

$$y(t) = -e^{-t}u(t) + 2e^{-2t}u(t) - e^{-3t}u(t). \tag{17}$$

Is this faster than using our formulas for $e^{-at}u(t) * e^{-bt}u(t)$? Maybe!

Another example

Suppose

$$x(t) = \cos(t)u(t) \tag{18}$$

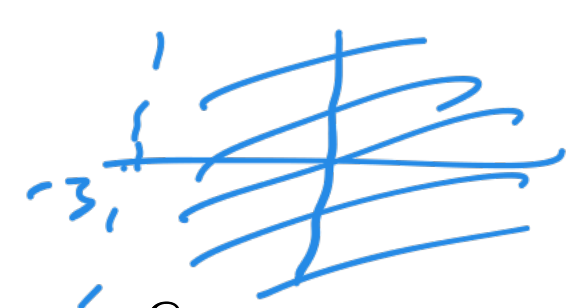
$$h(t) = e^{-3t}u(t) \tag{19}$$

We want to find $y(t)$ and $Y(s)$ and its ROC.

First step: take the Laplace transforms of $x(t)$ and $h(t)$. Using the transform table:

$\omega_0 = 1$

$$X(s) = \frac{s}{s^2+1} \qquad \boxed{\{\Re\mathfrak{e}\{s\} > 0\}}$$



$$H(s) = \frac{1}{s+3} \qquad \{\Re\mathfrak{e}\{s\} > -3\}$$

$$Y(s) = \frac{s}{(s^2+1)(s+3)} = \frac{s}{(s-j)(s+j)(s+3)} \qquad \boxed{\{\Re\mathfrak{e}\{s\} > 0\}}$$

Partial fraction expansion with imaginary roots

$$\left.\frac{s}{(s+j)(s+3)}\right|_{s=j} = \frac{j}{(2j)(j+3)} = \frac{1}{6+2j} = \alpha \tag{20}$$

$$\left.\frac{s}{(s-j)(s+3)}\right|_{s=-j} = \frac{-j}{(-2j)(-j+3)} = \frac{1}{6-2j} = \alpha^* \tag{21}$$

$$\left.\frac{s}{(s+j)(s-j)}\right|_{s=-3} = \frac{-3}{(-3+j)(-3-j)} = \frac{-3}{10} \tag{22}$$

So

$$Y(s) = \frac{\alpha}{s-j} + \frac{\alpha^*}{s+j} - \frac{3/10}{s+3} \tag{23}$$

$$= \frac{(\alpha+\alpha^*)s + (j\alpha - j\alpha^*)}{s^2+1} - \frac{3/10}{s+3} \tag{24}$$

Plugging in α

We have

$6^2 + 2^2 = 36 + 4 = 40$

$$\alpha = \frac{1}{6+2j} = \frac{6}{40} - \frac{2}{40}j = \frac{3}{20} - \frac{1}{20}j \tag{25}$$

$$\alpha + \alpha^* = \frac{6}{20} = \frac{3}{10} \quad = 2\,\mathrm{Re}\{\alpha\} \tag{26}$$

$$j\alpha - j\alpha^* = \frac{2}{20} = \frac{1}{10} \quad = 2\,\mathrm{Im}\{\alpha^*\} \tag{27}$$

That makes

cos sin e^{-3t}

$$Y(s) = \frac{(3/10)s + 1/10}{s^2+1} - \frac{3/10}{s+3} \tag{28}$$

Back to the time domain

$$Y(s) = \frac{(3/10)s + 1/10}{s^2 + 1} - \frac{3/10}{s + 3} \tag{29}$$

ROC $\{\mathrm{Re}\{s\} > 0\}$

Now use $\cos(t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2+1}$ and $\sin(t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s^2+1}$:

$$y(t) = \frac{3}{10}\cos(t)u(t) + \frac{1}{10}\sin(t)u(t) - \frac{3}{10}e^{-3t}u(t) \tag{30}$$

So what happened?

- The impulse response "mode" appears in the output.
- The cosine is phase shifted to give a cosine and sine part.

Try it yourself

Problem

Find the output time domain signals for the inputs $x(t)$ and impulse response $h(t)$:

- $x(t) = e^{-6t}u(t) + e^{-3t}u(t)$, $h(t) = e^{-4t}u(t)$.
- $x(t) = -e^{-2t}u(-t) + e^{-5t}u(t)$, $h(t) = e^{-3t}u(t)$.
- $x(t) = \sin(3t)u(t) + e^{4t}u(-t)$, $h(t) = e^{-t}u(t)$.
- $x(t) = e^{-t}\cos(t)u(t)$, $h(t) = e^{-2t}u(t)$.

Warning: for some of them you will have to do partial fraction expansion with complex s. The last one may be particularly messy.