Linear Systems and Signals
Laplace transform examples

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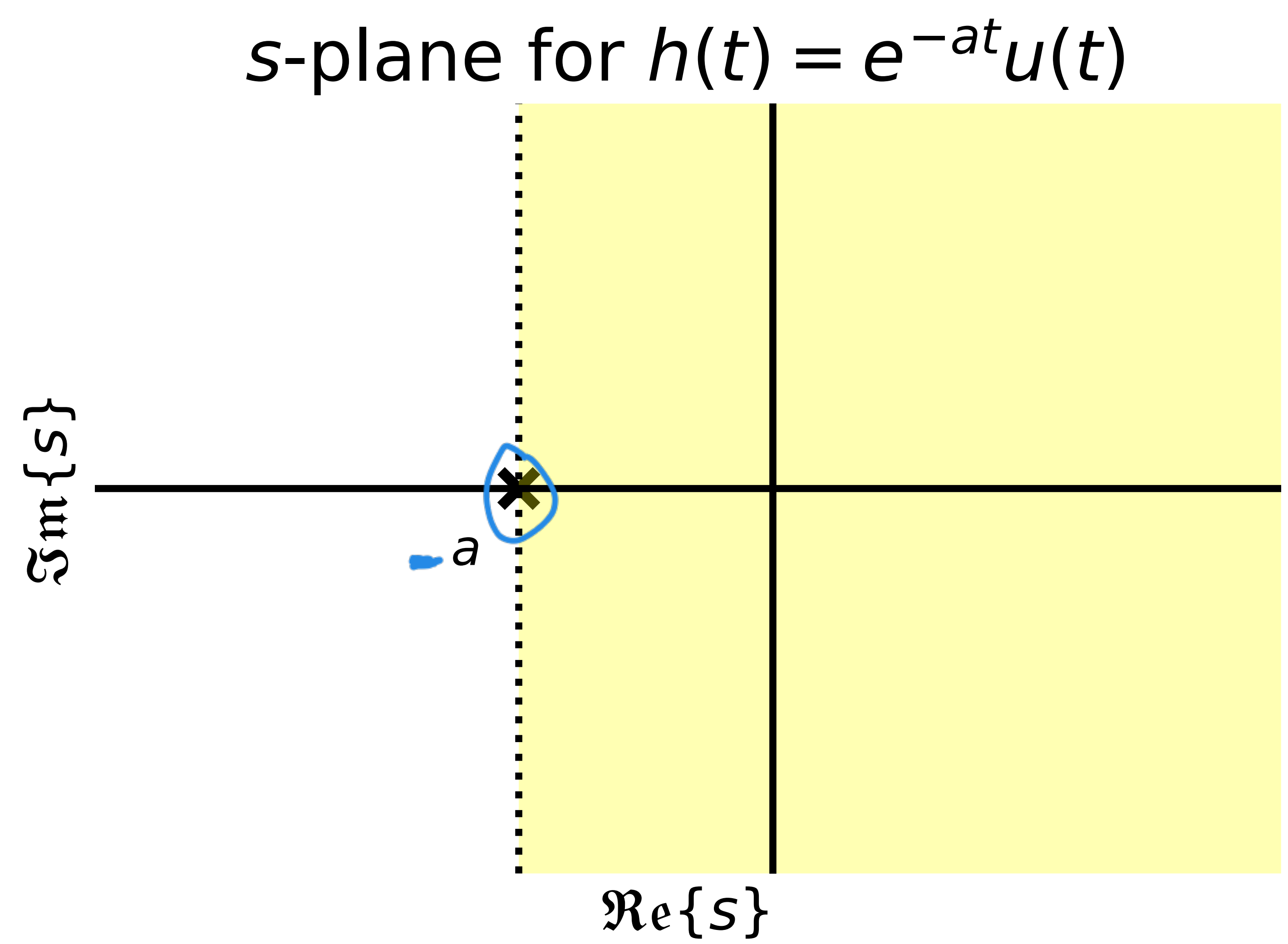
Learning objectives

The learning objectives for this section are:

- use the definition to compute Laplace transforms
- identify the region of convergence for a Laplace transform

General exponentials



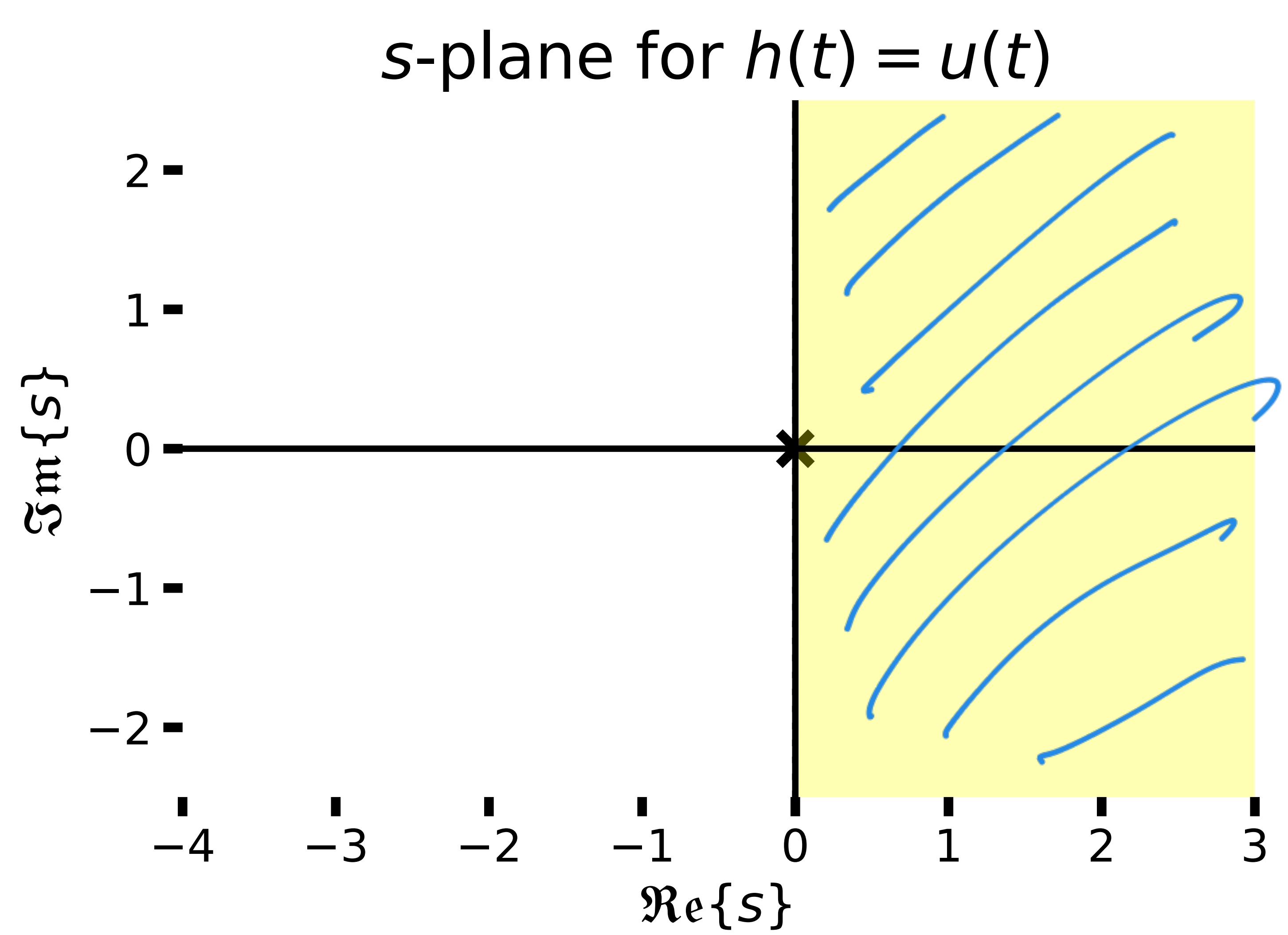
$\mathfrak{Re}\{s\} > -a$

If $h(t) = e^{-at}u(t)$ we get a general formula:

$$H(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-(s+a)t}dt = \frac{1}{s+a}. \qquad (1)$$

What about the ROC? We need $\mathfrak{Re}\{s\} > -a$. This has a *pole* at $-a$.

The unit step



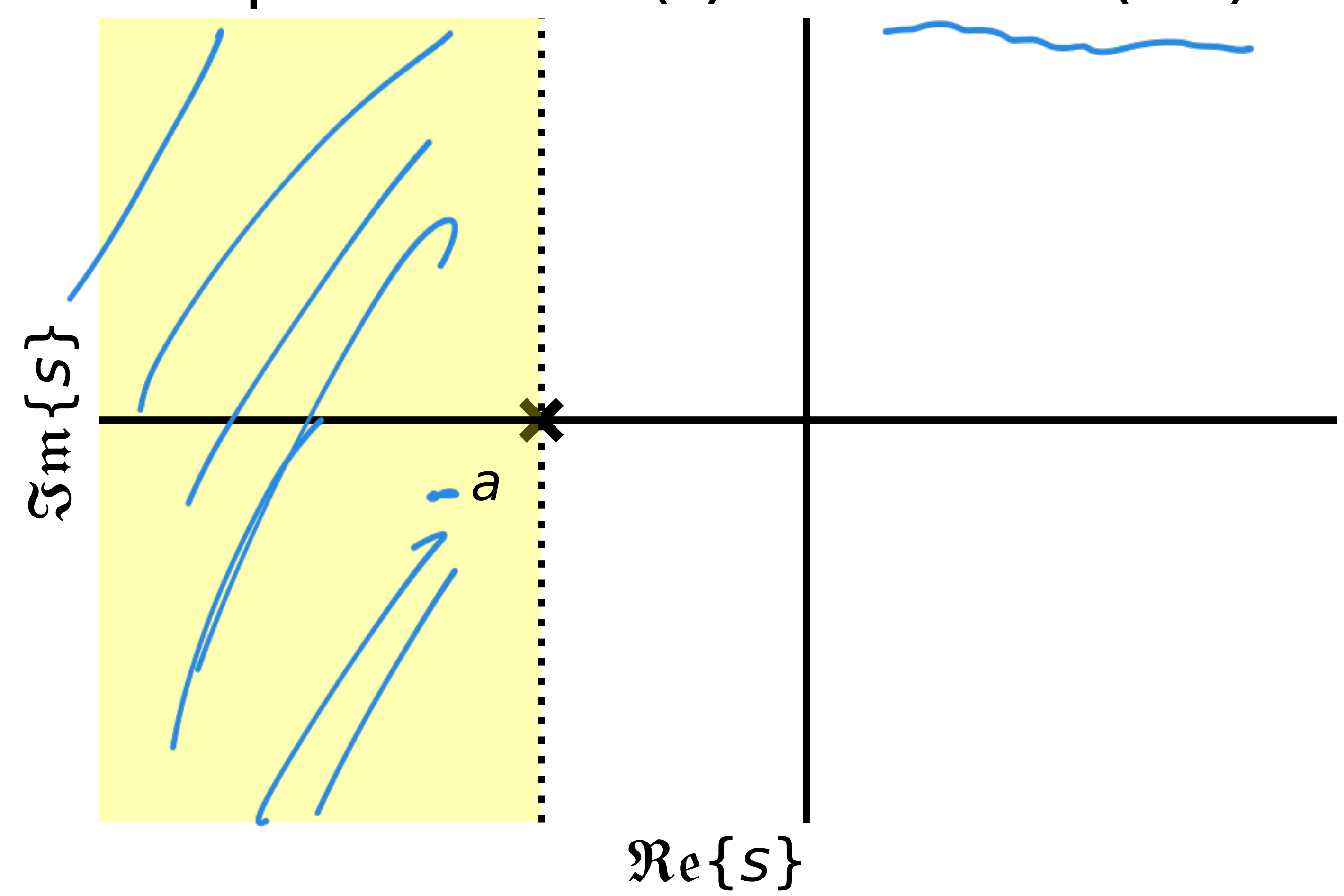
If $h(t) = u(t)$ then the system is an *integrator*:

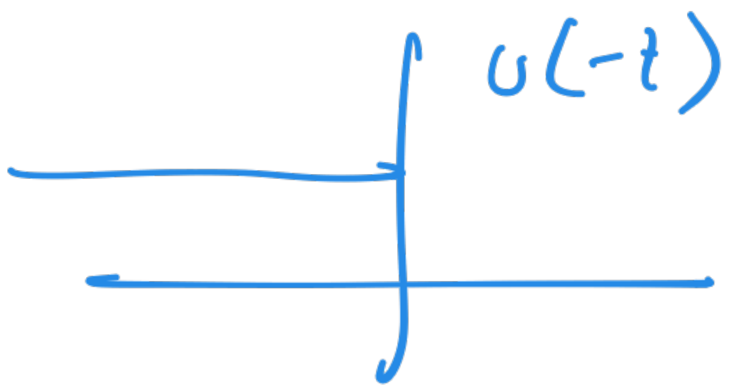
$$H(s) = \int_{-\infty}^{\infty} u(t) e^{-st} dt = \int_{0}^{\infty} e^{-st} dt = \frac{1}{s}. \quad (2)$$

What about the ROC? We need $\mathfrak{Re}\{s\} > 0$. We can also see this as the previous example with $a = 0$.

Anticausal systems

s-plane for $h(t) = -e^{-at}u(-t)$





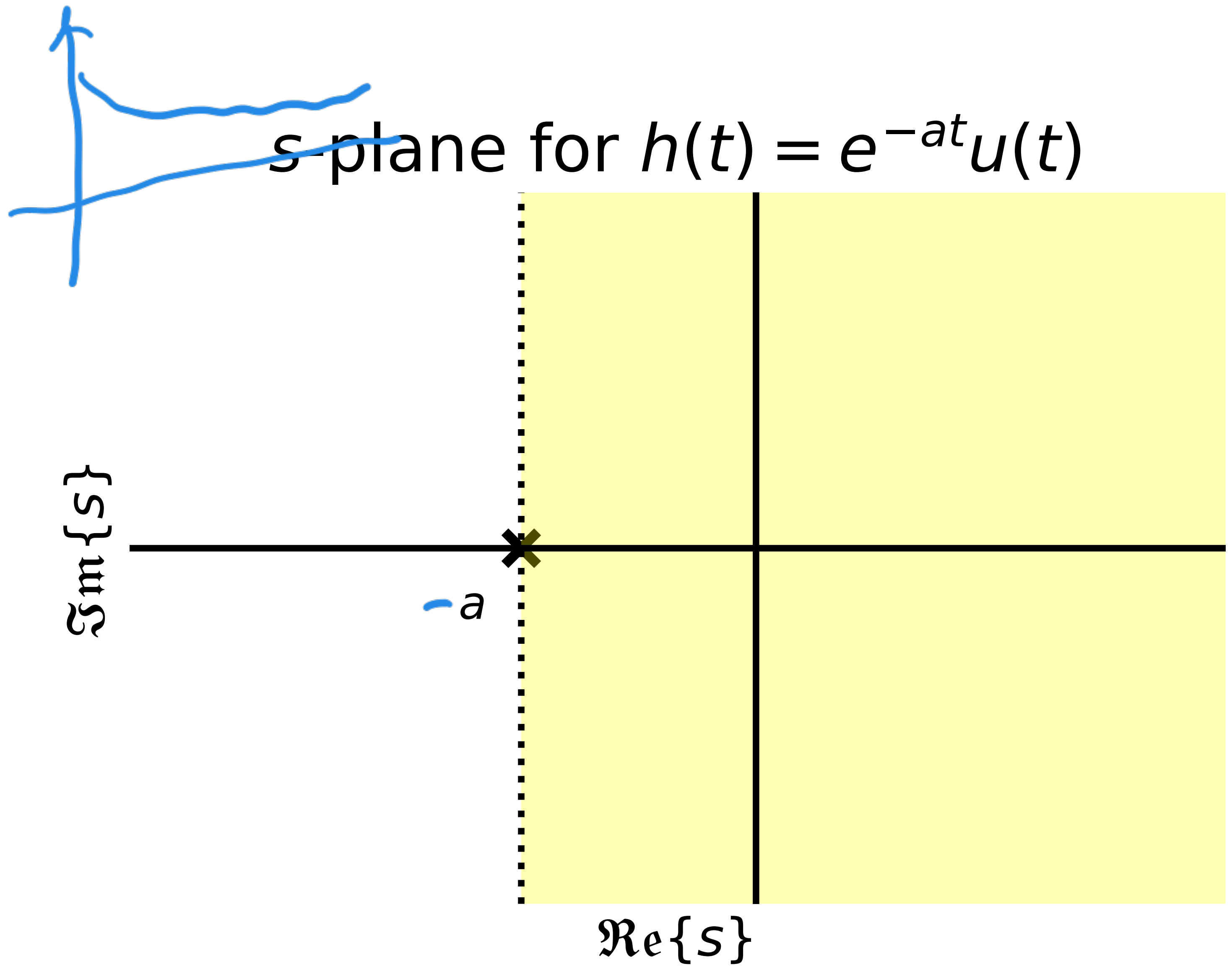
If $h(t) = -e^{-at}u(-t)$ we get a general formula:

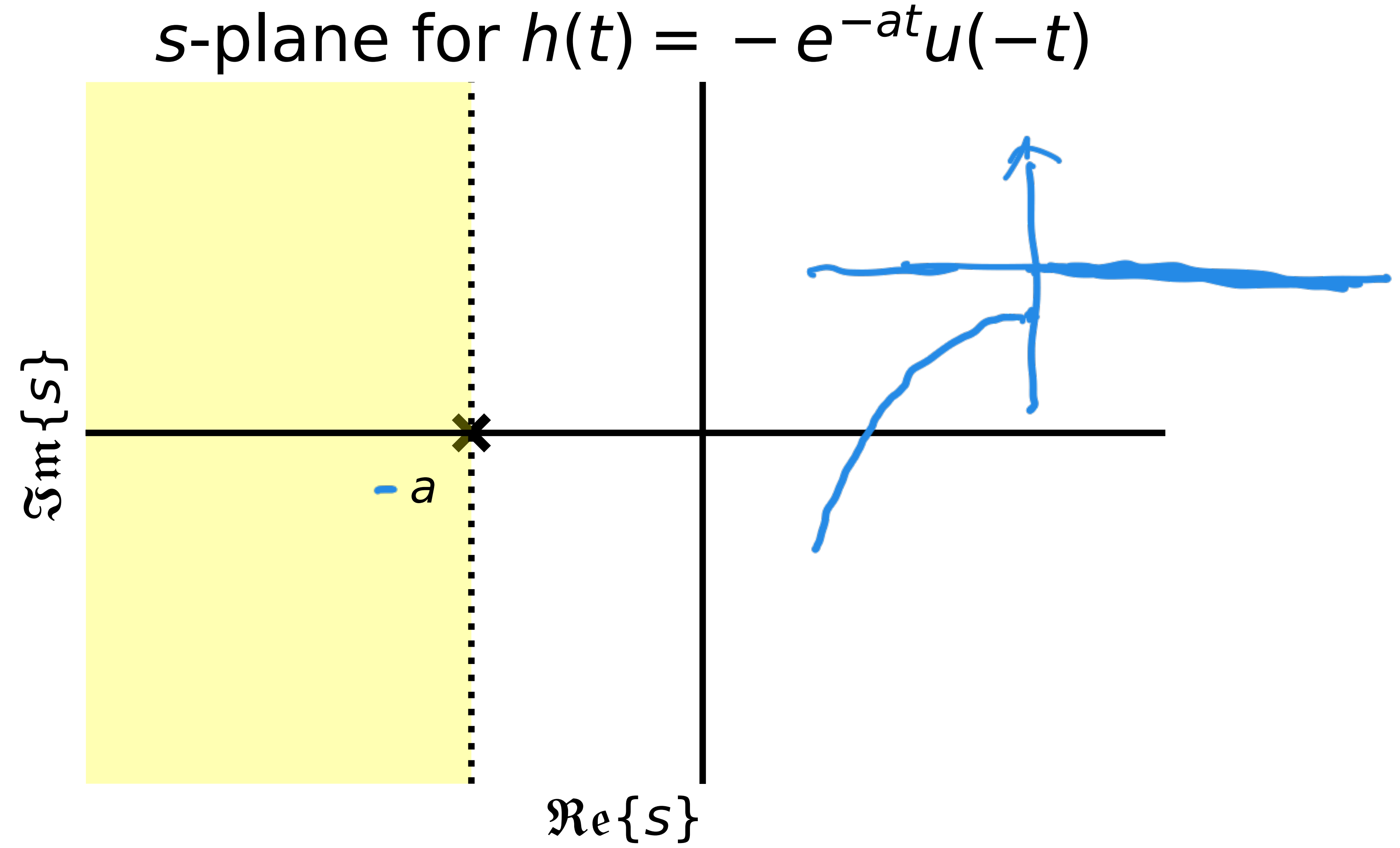
s+a < 0

$$H(s) = \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}dt = \int_{-\infty}^{\infty} -e^{-(s+a)t}dt = \boxed{\frac{1}{s+a}.} \quad (3)$$

What about the ROC? We need $\Re\mathfrak{e}\{s\} < -a$. This has a *pole* at $-a$ but the ROC goes the other way.

Two signals with the same transform?





We have both

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} \qquad\qquad -e^{-at}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} \tag{4}$$

The difference is that their ROCs are different. To write a Laplace transform, you always have to specify the ROC!

A cosine

Let's take $h(t) = \cos(\omega t) u(t)$. First step is to Eulerize:

$$\cos(\omega t) u(t) = \left(\frac{1}{2} e^{j\omega t} + \frac{1}{2} e^{-j\omega t} \right) u(t). \tag{5}$$

Taking the Laplace transform:

$$H(s) = \int_{-\infty}^{\infty} \left(\frac{1}{2} e^{j\omega t} u(t) + \frac{1}{2} e^{-j\omega t} \right) u(t) e^{-st} dt \tag{6}$$

$$= \frac{1}{2} \int_0^{\infty} e^{-(s-j\omega)t} dt + \frac{1}{2} \int_0^{\infty} e^{-(s+j\omega)t} dt \tag{7}$$

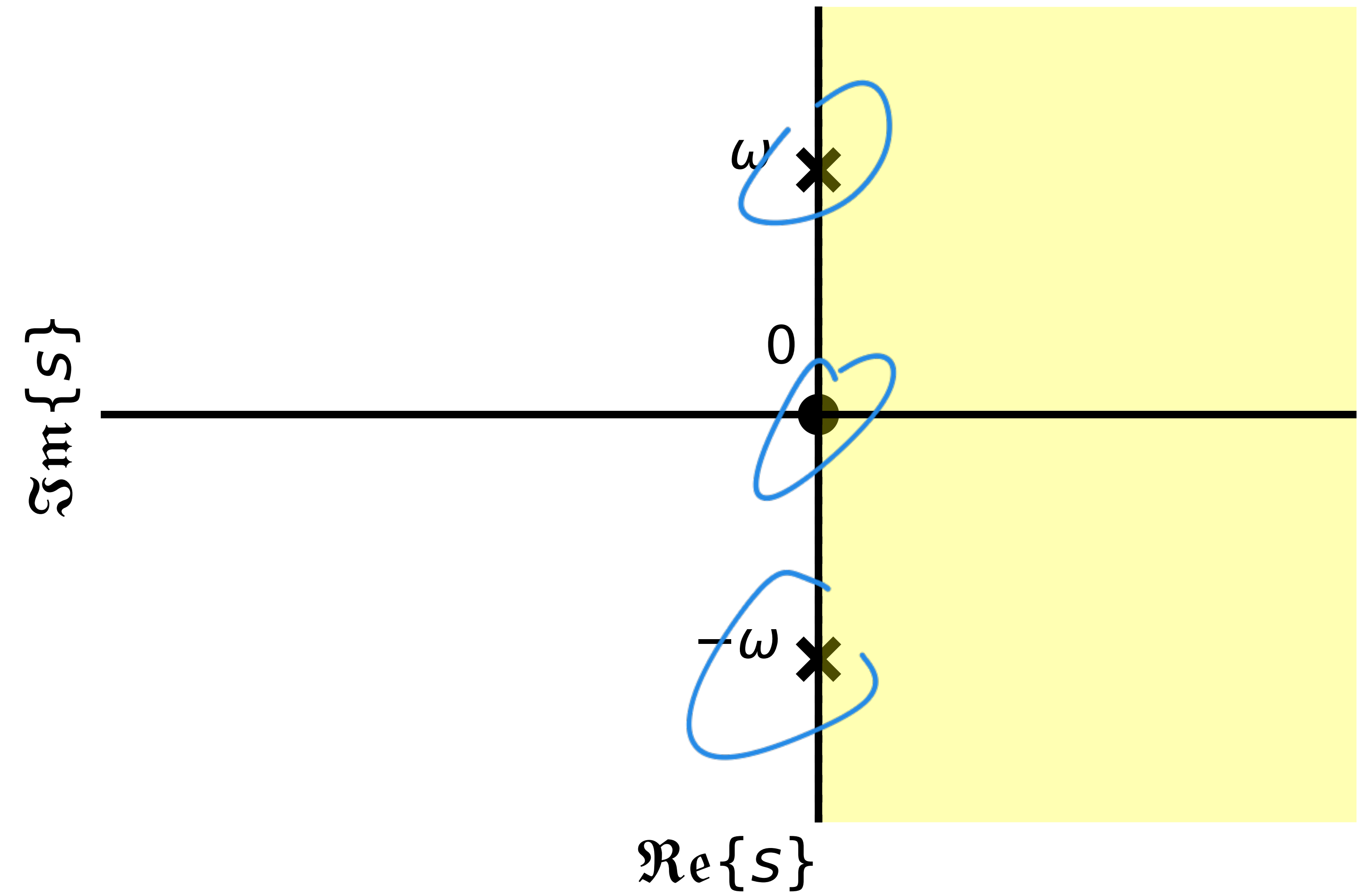
$$= \frac{1/2}{s - j\omega} + \frac{1/2}{s + j\omega} \tag{8}$$

$$\boxed{= \frac{s}{s^2 + \omega^2}.} \tag{9}$$

degree 2 : 2 roots
$\Rightarrow$ 2 poles

A cosine

s-plane for $h(t) = \cos(\omega t)u(t)$



pole-zero diagram

So we have

zero

$$\boxed{\cos(\omega t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2 + \omega^2}.} \tag{10}$$

This is zero at $s = 0$ and blows up (has poles) at $s = \pm j\omega$

Try it yourself

Problem

Try to find the following Laplace transforms from the definition, without using the Laplace Transform table. However, in practice you can use the table to just look up the relevant transforms.

- $e^{-2t}u(t) - e^{3t}u(-t)$
- $e^{-t}\cos(2\pi t)u(t)$
- $\sin(\omega t)u(t)$

Don't forget the region of convergence!