Linear Systems and Signals
Laplace transforms from differential equations

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Learning objectives

The learning objectives for this section are: LCCDEs

- convert linear constant coefficient differential equations into rational Laplace transforms
- find the impulse response of a rational LTI system
- use known transforms to simplify inverting Laplace transforms

LCCDEs

Possibly the first time you saw Laplace transforms was in a class on differential equations. Linear constant coefficient differential equations (LCCDEs) give us rational Laplace transforms using the property $\boxed{\frac{d}{dt}x(t) \overset{\mathcal{L}}{\longleftrightarrow} sX(s).}$ Suppose we have a system with input-output relation

$$5\frac{d^2}{dt^2}y(t) - 3\frac{d}{dt}y(t) + y(t) = -\frac{d}{dt}x(t) + \frac{1}{2}x(t) \qquad (1)$$

then taking Laplace transforms on both sides:

$$(5s^2 - 3s + 1)Y(s) = \left(-s + \frac{1}{2}\right)X(s) \qquad (2)$$

$\frac{d}{dt} \to s$

$x(t) \to X(s)$

$y(t) \to Y(s)$

so the *transfer function is*

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1/2 - s}{5s^2 - 3s + 1}. \qquad (3)$$

From LCCDEs to rational transforms

In general, if you have system described by

$$b_0 y(t) + b_1 \frac{d}{dt} y(t) + b_2 \frac{d^2}{dt^2} y(t) + \cdots + b_\ell \frac{d^\ell}{dt^\ell} y(t) \tag{4}$$

$$= a_0 x(t) + a_1 \frac{d}{dt} x(t) + a_2 \frac{d^2}{dt^2} x(t) + \cdots + a_k \frac{d^k}{dt^k} x(t) \tag{5}$$

then the transfer function is a rational Laplace transform:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{a_k s^k + a_{k-1} s^{k-1} + \cdots + a_1 s + a_0}{b_\ell s^\ell + b_{\ell-1} s^{\ell-1} + \cdots + b_1 s + b_0} \tag{6}$$

$$= C \frac{(s - \alpha_1)(s - \alpha_1) \cdots (s - \alpha_k)}{(s - \beta_1)(s - \beta_1) \cdots (s - \beta_\ell)} \tag{7}$$

with zeros at $\alpha_1, \alpha_2, \ldots \alpha_k$ and poles at $\beta_1, \beta_2, \ldots, \beta_\ell$.

Example: getting the transfer function

Let's look at a *causal* system defined by the following LCCDE:

$$\frac{d^2}{dt^2}y(t) + 6\frac{d}{dt}y(t) + 13y(t) = \frac{d}{dt}x(t) - x(t) \tag{8}$$

then we have

$$(s^2 + 6s + 13)Y(s) = (s - 1)X(s) \tag{9}$$

so

$$\boxed{H(s) = \frac{Y(s)}{X(s)} = \frac{s-1}{s^2 + 6s + 13}} \tag{10}$$

Example: finding poles and zeros

zero @ s=1

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s-1}{s^2+6s+13} \tag{11}$$

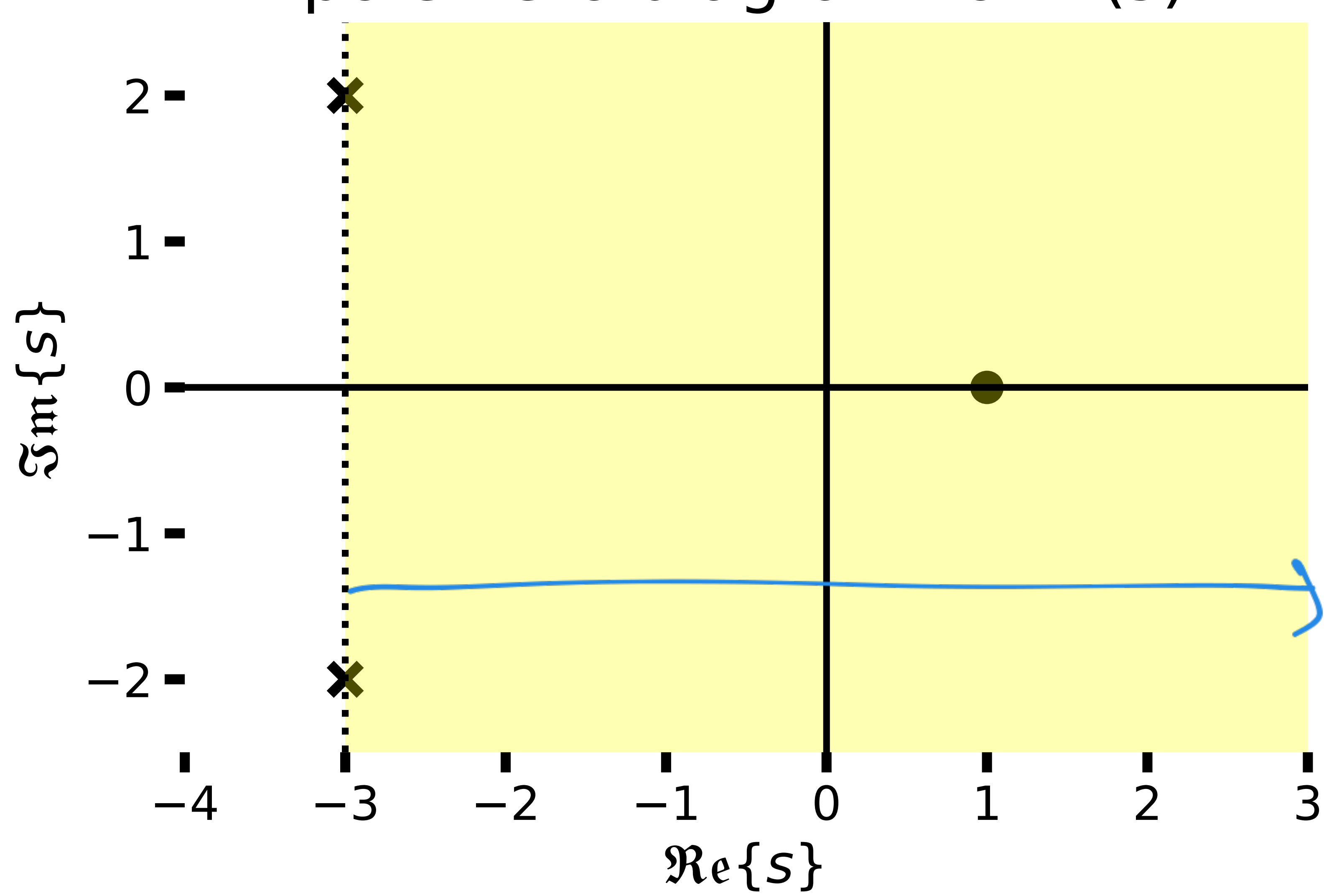
quad. form

There is a zero at $s = 1$ and poles at

4

$$\frac{-6 \pm \sqrt{36-52}}{2} = -3 \pm 2j \tag{12}$$

pole-zero diagram for $H(s)$

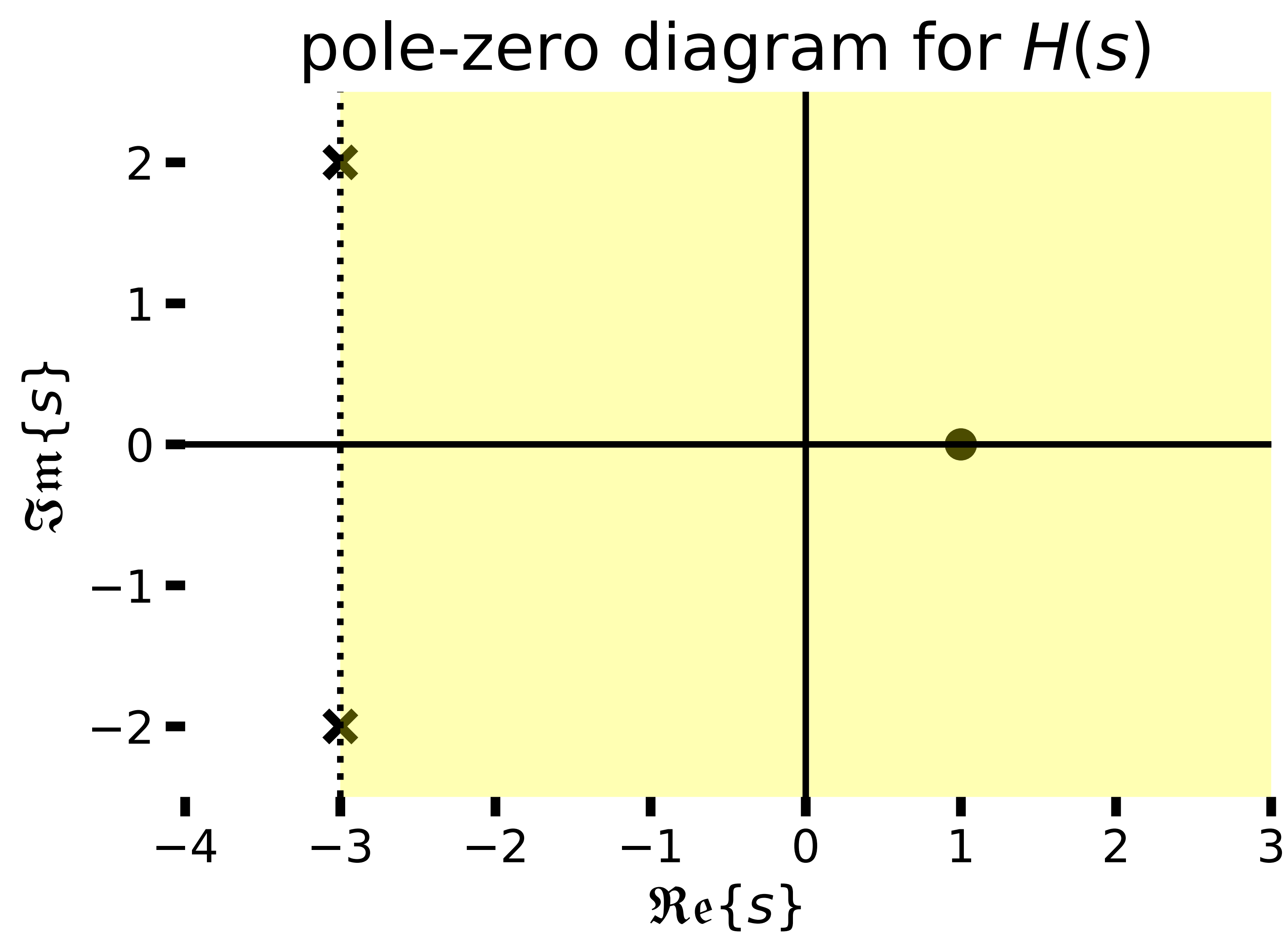


h(t) causal

→ ROC is a right half-plane

Example: using Laplace transform properties

conjugate pair of roots



We know the system is causal so the impulse response has to be right-sided (and starting at 0 or positive t). How do we get back from the poles and zeros? Maybe we can use Laplace transform properties. Look at the denominator:

$$H(s) \to \frac{s-1}{s^2+6s+13} = \frac{s-1}{(s+3)^2+4} \tag{13}$$

Example: finding the time domain signals

We need transform pairs – derive the second in the same way we did with $e^{-at}\sin(\omega_0 t)u(t)$:

$$e^{-at}\sin(\omega_0 t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{\omega_0}{(s+a)^2+\omega_0^2} \tag{14}$$

$$e^{-at}\,\cancel{e^{-at}}\cos(\omega_0 t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s+a}{(s+a)^2+\omega_0^2} \tag{15}$$

Setting $\omega_0 = 2$ and $a = 3$: $\omega_0^2 = 4 \Rightarrow \omega_0 = 2$

$$\frac{s-1}{(s+3)^2+4} = \frac{s+3}{(s+3)^2+4} - \frac{4}{(s+3)^2+4} \tag{16}$$

$$\overset{\mathcal{L}}{\longleftrightarrow} e^{-3t}\cos(2t)u(t) - 2e^{-3t}\sin(2t)u(t). \tag{17}$$

So $h(t) = e^{-3t}\cos(2t)u(t) - 2e^{-3t}\sin(2t)u(t)$.

Looking back

The steps in finding the impulse response from an LCCDE:

1. Convert all $\frac{d}{dt}$ operations into multiplication by s.
2. Get the Laplace transform $H(s) = \frac{Y(s)}{X(s)}$. ← transfer function
3. Find the poles and zeros and plot the pole-zero diagram. ← choose an ROC
4. Invert the rational transform using partial fraction expansion or /and known transform pairs.

Try it yourself

Problem

Find the impulse response of the following systems defined by LCCDEs. Assume that all impulse responses are causal.

$$\frac{d^2}{dt^2}y(t) + 7\frac{d}{dt}y(t) + 12y(t) = x(t) \tag{18}$$

$$\frac{d^2}{dt^2}y(t) + 8\frac{d}{dt}y(t) + 12y(t) = -\frac{d}{dt}x(t) \tag{19}$$

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 8y(t) = \frac{d}{dt}x(t) + 5x(t) \tag{20}$$