Linear Systems and Signals
Introduction to Laplace Transforms

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Learning objectives

The learning objective for this section is:

- apply the definition of the Laplace transform

The eigenfunction property

$$Ce^{j\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow H(j\omega_0)Ce^{j\omega_0 t}$$

e^{st} $\cdot\ e^{st}$

Remember the eigenfunction property of LTI systems:

complex coefficient

$$h(t) * e^{j\omega_0 t} = H(j\omega_0)e^{j\omega_0 t}. \tag{1}$$

If you put in a complex exponential into a CT LTI system you will get the same complex exponential out, but with a complex scaling $H(j\omega_0)$.

What happens if we we replace $j\omega_0$ with an arbitrary complex number s?

The Laplace Transform

$$h(t) * e^{st} = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau \qquad (2)$$

Definition

The (bilateral) Laplace transform of a signal $h(t)$ is defined as

$$\boxed{H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau.} \qquad (3)$$

for all values of s where the integral converges.

We will use capital letters for transforms and write $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$, etc.

Why bilateral?

$$\int_{-\infty}^{\infty} h(t)\,u(t)\,e^{-st}\,dt = \int_{0}^{\infty} h(t)\,e^{-st}\,dt$$

unilateral

$$H(s) = \int_{\cancel{-\infty}\ 0}^{\infty} h(t)e^{-st}dt. \tag{4}$$

- We call it *bilateral* because t goes from $-\infty$ to ∞. The so-called *unilateral* transform goes from 0 to ∞.
- If we only consider *causal* systems, then we can think of $h(t)$ as $\tilde{h}(t)u(t)$ since the impulse response has to be 0 for $t < 0$.
- The *unilateral* transform is useful if we are talking about causal systems only.

Convergence and regions thereof

In the definition we said "for all values of s where the integral converges." What do we mean by that? Take $h(t) = e^{-2t}u(t)$:

$$H(s) = \int_{-\infty}^{\infty} e^{-2t}u(t)e^{-st}dt \tag{5}$$

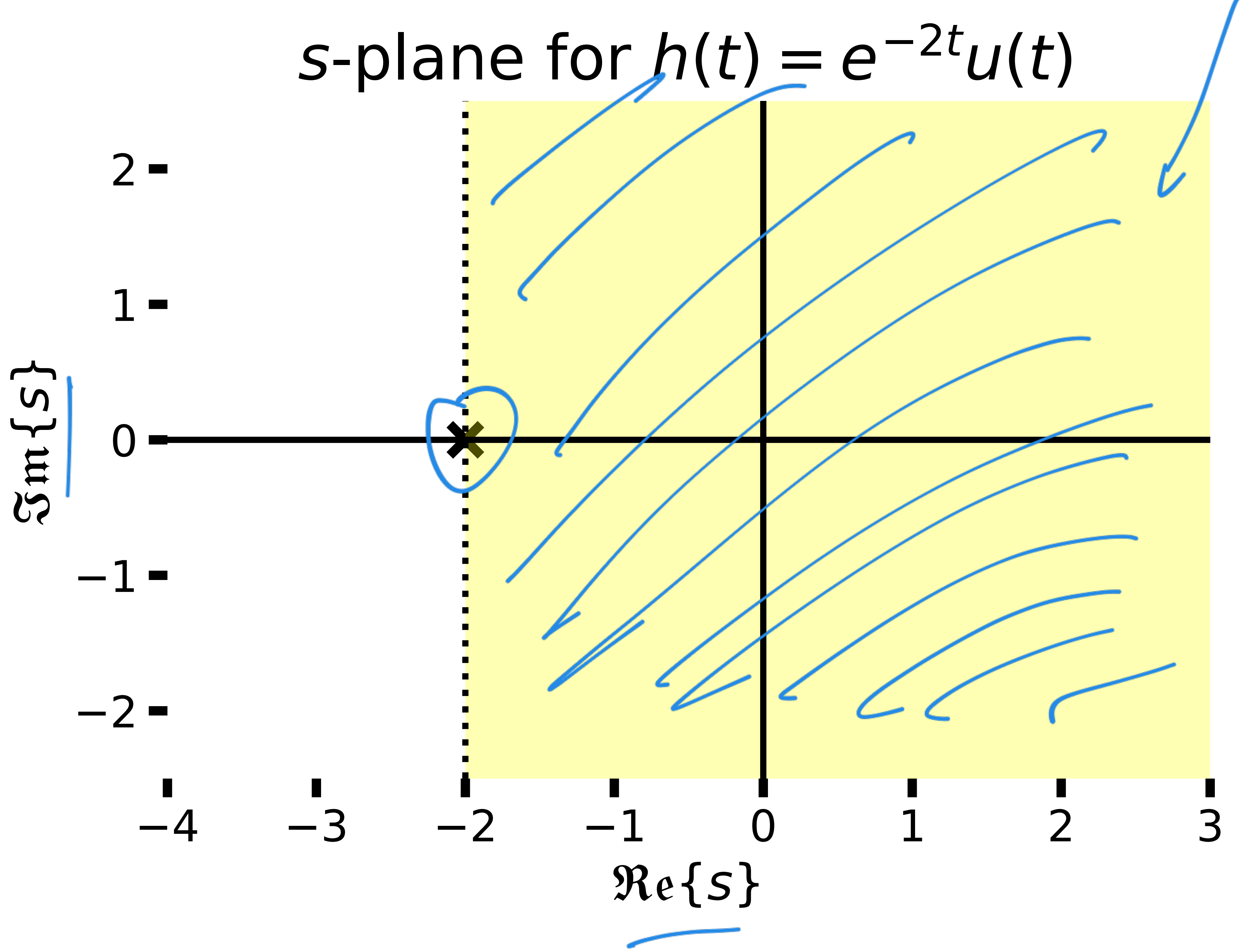
$$= \int_{0}^{\infty} \boxed{e^{-(2+s)t}}\, dt \tag{6}$$

if this is negative, integral diverges!

$$= \frac{1}{2+s}. \tag{7}$$

The last line only works as long as $2 + s > 0$ or $s > -2$. What we actually need is $\Re\mathfrak{e}\{s\} > -2$. The set $\boxed{\{s : \Re\mathfrak{e}\{s\} > -2\}}$ is called the *region of convergence (ROC)* of the transform.

The s-plane



s is complex

H(s) is complex

s-plane is $\mathbb{C}$

We can think of the Laplace transform as a complex valued function $H : \mathbb{C} \to \mathbb{C}$. The domain of H is called the s-plane. For $H(s) = \frac{1}{s+2}$ we see that there is a *pole* at $s = -2$ marked with an $\times$.

Another example

We can use the sifting property of the Dirac δ function. The identity system has impulse response of $\delta(t)$:

$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$

$$\delta(t) \overset{\mathcal{L}}{\leftrightarrow} \int_{-\infty}^{\infty} \underbrace{\delta(t)}_{h(t)} e^{-st} dt = 1. \quad (8)$$

$t=0$

$\delta(t) \overset{\mathcal{L}}{\longleftrightarrow} 1$

$ROC = \mathbb{C}$

What about a system that delays its input by τ?

$t=\tau$

$$\delta(t-\tau) \overset{\mathcal{L}}{\leftrightarrow} \int_{-\infty}^{\infty} \underline{\delta(t-\tau)} e^{-st} dt = \boxed{e^{-s\tau}.} \quad (9)$$

$\delta(t-\tau) \overset{\mathcal{L}}{\longleftrightarrow} e^{-s\tau}$

delay by τ

We will see other examples in the next sections.

Coming up next

In the next few presentations we will dig a little deeper:

- examples of calculating the Laplace transform
- describing the ROC and its properties
- showing how convolution and Laplace transforms are related

$h(t) * x(t) \overset{\mathcal{L}}{\longleftrightarrow} H(s)X(s)$

A critical fact to remember is that a Laplace transform of a signal is not just the formula – you need to describe the region of conference (ROC) as well.