Linear Systems and Signals

Laplace transform properties, part 1

Anand D. Sarwate

Department of Electrical and Computer Engineering
Rutgers, The State University of New Jersey

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Learning objectives

The learning objectives for this section are:

- understand properties of the Laplace transform
- use transform properties to find Laplace transforms

We will use

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \qquad \text{or} \qquad \mathcal{L}\{x(t)\} = X(s) \tag{1}$$

and label the ROC $\mathcal{R}$.

The Laplace transform is linear

$$\mathcal{L}(x(t)) = \boxed{X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt} \tag{2}$$

We already saw that the Laplace transform is linear. If

$$x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s) \qquad x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s) \tag{3}$$

with ROCs $\mathcal{R}_1$ and $\mathcal{R}_2$, then for scalars a_1 and a_2,

$$a_1 x_1(t) + a_2 x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} a_1 X_1(s) + a_2 X_2(s). \tag{4}$$

The ROC contains $\mathcal{R}_1 \cap \mathcal{R}_2$.

Time shift

What if we delay $x(t)$ by t_0?

$\tau = t - t_0$

$$x(t-t_0) \overset{\mathcal{L}}{\longleftrightarrow} \int_{-\infty}^{\infty} x(t-t_0)e^{-st}dt = \int_{-\infty}^{\infty} x(\tau)e^{-s(\tau+t_0)}d\tau \qquad (5)$$

$$= e^{-st_0}X(s) \qquad (6)$$

delayed version of x

With the same ROC. It turns out we have the Laplace transform pair

$$\delta(t) \overset{\mathcal{L}}{\longleftrightarrow} 1 \qquad (7)$$

Which means

LTI system impulse response that delays the input by t_0

$$\delta(t-t_0) \overset{\mathcal{L}}{\longleftrightarrow} e^{-st_0}. \qquad (8)$$

In words: a system that delays by t_0 has a Laplace transform e^{-st_0}.

Example

Find the Laplace transform of

cosine! delay

$$x(t) = 3\cos(4\pi t)u(t) + e^{-3t}u(t-3) \tag{9}$$

We can use linearity and time shifts:

$$x(t) = 3\mathcal{L}\{\cos(4\pi t)u(t)\} + \mathcal{L}\left\{e^{-9}e^{-3(t-3)}u(t-3)\right\} \tag{10}$$

$$= \frac{3s}{s^2 + (4\pi)^2} + \frac{e^{-9}e^{-3s}}{s+3}. \tag{11}$$

e^{-st_0} $t_0 = 3$

$e^{-3t}u(t)$

The ROC is $\{s : \Re\{s\} > 0\} \cap \{s : \Re\{s\} > -3\} = \boxed{\{s : \Re\{s\} > 0\}}$.

s-domain shift

What about a shift in s instead of a shift in time? Using the same type of argument as time shifts, we get

$$e^{s_0 t} x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s - s_0) \tag{12}$$

Now the ROC shifts by $\mathfrak{Re}\{s_0\}$: if $s_0 > 0$ it shifts to the right, and if $s_0 < 0$ it shifts to the left.

Example

Find the Laplace transform of

decaying exp.
$s_0 = -a$

$$x(t) = e^{-at} \sin(\omega_0 t) u(t). \quad (13)$$

We have

$$\sin(\omega_0 t) u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{\omega_0}{s^2 + \omega_0^2}. \quad (14)$$

shift this
X(s) by $-a$

So

$$e^{-at} \sin(\omega_0 t) u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{\omega_0}{(s+a)^2 + \omega_0^2} \quad (15)$$

where are
the poles?

with ROC $\{\mathfrak{Re}\{s\} > -a\}$ (shifted to the left by a).

Time dilation/compression

If we look at $x(at)$, we can plug into the transform to get

$$x(at) \stackrel{\mathcal{L}}{\longleftrightarrow} \int_{-\infty}^{\infty} x(at)e^{-st} = \int_{-\infty}^{\infty} \frac{1}{|a|} x(\tau)e^{-(s/a)\tau} d\tau \tag{16}$$

$$= \frac{1}{|a|} X\left(\frac{s}{a}\right) \tag{17}$$

The ROC is:

$$\mathcal{R}(x(at)) = \{s : s/a \in \mathcal{R}\}. \tag{18}$$

If $|a| < 1$ we are slowing down $x(t)$ and the ROC shrinks by a factor of a and if $|a| > 1$ we are speeding up $x(t)$ and the ROC expands by a factor of a.

Conjugation

If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$ then

$$x^*(t) \overset{\mathcal{L}}{\leftrightarrow} = \int_{-\infty}^{\infty} x^*(t) e^{-st} dt = \left(\int_{-\infty}^{\infty} x(t) e^{-s^* t} \right)^* \tag{19}$$

$$= X^*(s^*). \tag{20}$$

$X(s^*)$

Why is this useful? If $x(t)$ is real, then $x(t) = x^*(t)$ and $X(s) = X^*(s^*)$. That means if $X(s)$ has a pole or zero at $s = s_0$, then it also has one at $s = s_0^*$.

We say that for real signals, the poles and zeros appear in *conjugate pairs*.

Try it yourself

Problem

Try finding the following Laplace transforms using the properties:

$$x(t) = \delta(t-2) - 3e^{-5t}u(t) \tag{21}$$

$$x(t) = e^{-3t}u(t) + e^{-4t}\cos(6t)u(t) \tag{22}$$

$$x(t) = e^{-2|t|} \tag{23}$$