Linear Systems and Signals

Laplace transform properties, part 2

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Learning objectives

The learning objectives for this section are:

- understand properties of the Laplace transform
- use transform properties to find Laplace transforms

Convolution property

The most important property/fact about the Laplace transform is "convolution in the time domain is multiplication in the Laplace domain."

Theorem

Let $x(t)$ and $h(t)$ be two CT signals with Laplace transforms $H(s)$ with ROC $\mathcal{R}_h$ and $X(s)$ with ROC $\mathcal{R}_x$. Then the following Laplace transform pair holds:

$$\overset{\text{time domain}}{\underline{(h * x)(t)}} \overset{\mathcal{L}}{\longleftrightarrow} \overset{\text{Laplace domain}}{\underline{H(s)X(s)}}, \tag{1}$$

where the ROC of $\underline{H(s)X(s)}$ contains $\underline{\mathcal{R}_x \cap \mathcal{R}_h}$.

Proving the most important property

def of convolution

def of LT

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \right) e^{-st} dt \tag{2}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} x(t-\tau) e^{-s(t-\tau)} d\tau dt \tag{3}$$

$$= \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} \int_{-\infty}^{\infty} x(t-\tau) e^{-s(t-\tau)} dt d\tau \tag{4}$$

LT of x

$$= \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} X(s) d\tau \tag{5}$$

LT of h

$$= H(s)X(s). \tag{6}$$

What does it mean?

This property means that we can find the outputs of LTI systems by multiplying the corresponding Laplace transforms and then inverting the Laplace transform. Suppose $x(t) = e^{-3t}u(t)$ and $h(t) = e^{-t}u(t)$. Then

$$y(t) = (h * x)(t) \overset{\mathcal{L}}{\longleftrightarrow} Y(s) = \frac{1}{s+3} \cdot \frac{1}{s+1}. = \frac{1}{(s+3)(s+1)} \tag{7}$$

How do we invert the Laplace transform? *Partial fraction expansion*.

$$\left.\frac{\cancel{(s+1)}}{(s+3)\cancel{(s+1)}}\right|_{s=-1} = \frac{1}{2} \qquad \left.\frac{\cancel{(s+3)}}{\cancel{(s+3)}(s+1)}\right|_{s=-3} = -\frac{1}{2} \tag{8}$$

So we have:

$$Y(s) = \frac{-1/2}{s+3} + \frac{1/2}{s+1} \overset{\mathcal{L}}{\longleftrightarrow} -\frac{1}{2}e^{-3t}u(t) + \frac{1}{2}e^{-t}u(t) \tag{9}$$

Differentiation in time or frequency

We have the following two properties:

$$\frac{d}{dt}x(t) \overset{\mathcal{L}}{\longleftrightarrow} sX(s) \qquad -tx(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{d}{ds}X(s), \tag{10}$$

where the ROC is the same $\mathcal{R}$ as $X(s)$.

Example: Find the Laplace transform of $te^{-at}u(t)$:

$$\text{BAD:} \qquad X(s) = \int_0^{\infty} te^{-at}e^{-st}dt \tag{11}$$

integration by parts.

$$\text{GOOD:} \qquad X(s) = -\frac{d}{ds}\left(\frac{1}{s+a}\right) = \frac{1}{(s+a)^2}. \tag{12}$$

Linear constant coefficient diff. eq.

Linear constant coefficient differential equations (LCCDEs) are often where you first encounter the Laplace transform:

$$2x(t) = \frac{d^2}{dt^2}x(t) + 2\frac{d}{dt}x(t) \tag{13}$$

$$\overset{\mathcal{L}}{\longleftrightarrow} \tag{14}$$

$$2X(s) = s^2X(s) + 2sX(s) \tag{15}$$

$$X(s) = \frac{1}{s^2 + 2s - 2} = \frac{1}{(s+2)(s-1)} \tag{16}$$

Then you can use partial fraction expansion to find $x(t)$ (try it!).

Integration in time

integrator

h(t) = u(t)

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}X(s) \tag{17}$$

$\frac{1}{s}$

The ROC contains $\mathcal{R} \cap \{s : \mathfrak{Re}\{s\} > 0\}$. There is a pole at $s = 0$ so the ROC cannot contain the imaginary axis.

Suppose $h(t) = u(t)$. Then for any $x(t)$,

$$(h * x)(t) = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau == \int_{-\infty}^{t} x(\tau)d\tau. \tag{18}$$

Since $(h * x)(t) \overset{\mathcal{L}}{\longleftrightarrow} H(s)X(s)$ this shows that

$$H(s) = U(s) = \frac{1}{s}. \tag{19}$$

Initial and final value theorems

If $x(t) = 0$ for $t < 0$ and has no δ-functions or other singularities at $t = 0$, then

$$\lim_{t \downarrow 0} x(t) = x(0^+) = \lim_{s \to \infty} sX(s). \tag{20}$$

If $x(t)$ has a finite limit as $t \to \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s) \tag{21}$$

These are mostly useful in some problems with rational Laplace transforms.

Try it yourself

Problem

Find the Laplace transforms, the ROC, and draw the pole-zero diagram for the following signals:

$$x(t) = 2\cos^2(t) + e^{-2t}u(t) \tag{22}$$

$$h(t) = t^2 e^{-3t} u(t) \tag{23}$$

$$y(t) = (te^{-t}u(t)) * (e^{-5t}\cos(5t)u(t)) \tag{24}$$

Find the time domain signals corresponding to the following Laplace transforms:

$$H(s) = \frac{s+3}{s^2+5s+4}, \mathcal{R} = \{\mathfrak{Re}\{s\} > -1\} \tag{25}$$

$$Y(s) = \frac{1}{s(s+2)}, \mathcal{R} = \{\mathfrak{Re}\{s\} > 0\} \tag{26}$$