Linear Systems and Signals

Finding the impulse response from an input-output relation

Anand D. Sarwate

Department of Electrical and Computer Engineering
Rutgers, The State University of New Jersey

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Learning objectives

The learning objectives for this section are:

- find the impulse response of a CT system from an input-output formula
- explain when one might need to characterize systems from I/O relations

Finding the impulse response from general input/output pa

Suppose I tell you that an LTI system with impulse response $h(t)$ has the output $y(t)$ when the input is $x(t)$:

suspect: h(t) has a single poles

$$\checkmark\; x(t) = e^{-3t}u(t) \tag{1}$$

$$\checkmark\; y(t) = -e^{-3t}u(t) + e^{-2t}u(t) \quad \approx \frac{\bullet}{s+2} \tag{2}$$

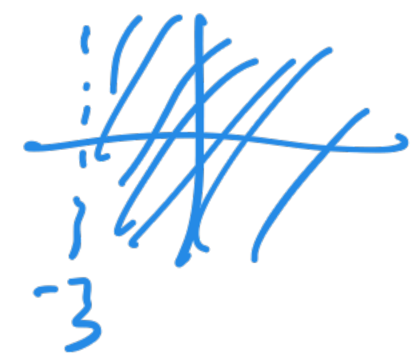
(guess)

To find $h(t)$ we can first find $H(s)$ from $X(s)$ and $Y(s)$ and then inverting.

Taking transforms on both sides

We have

$e^{-3t}u(t)$



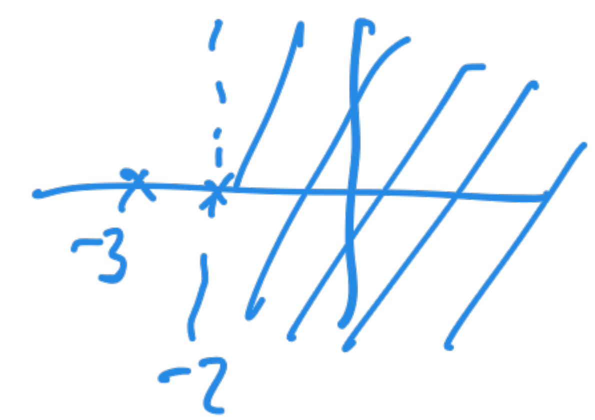
right-sided

$$X(s) = \frac{1}{s+3} \qquad \mathcal{R}_x = \{\Re\mathfrak{e}\{s\} > -3\} \qquad (3)$$

$$Y(s) = \frac{-1}{s+3} + \frac{1}{s+2} = \frac{1}{(s+2)(s+3)} \qquad \mathcal{R}_y = \{\Re\mathfrak{e}\{s\} > -2\} \qquad (4)$$

$-e^{-3t}u(t) + e^{-2t}u(t)$

So



$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{1}{(s+2)(s+3)}}{\frac{1}{s+3}} \qquad (5)$$

$$= \frac{1}{s+2} \qquad \boxed{\mathcal{R}_h = \{\Re\mathfrak{e}\{s\} > -2\}} \qquad (6)$$

$$\boxed{h(t) = e^{-2t}u(t).} \qquad (7)$$

ensure consistency

So we can find $h(t)$ given just one input-output pair $(x(t), y(t))$.

Step-by-step: system ID/characterization

Given $x(t)$ and $y(t)$, we can do the following:

1. Find $X(s)$ and $Y(s)$ with ROCs $\mathcal{R}_x$ and $\mathcal{R}_y$. need the ROC's
2. Calculate $H(s) = \frac{Y(s)}{X(s)}$ and an "appropriate" ROC $\mathcal{R}_h$. transfer function consistent $\mathcal{R}_h \cap \mathcal{R}_x = \mathcal{R}_y$
3. Use partial fraction expansion or long division plus inverse transforms to get $h(t)$.

Why is this important? If you have an *unknown* system, you can probe it with an input $x(t)$, measure $y(t)$, and then try to estimate the impulse response/transfer function. This is known as *system identification* and is very important when trying to characterize systems which you *don't* get to design.

Example 1: two-pole system

Suppose we have the following input/output pair:

$$x(t) = e^{-t}u(t) \tag{8}$$

3 terms!

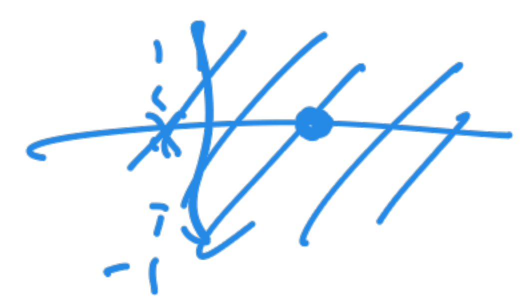
$$y(t) = \frac{1}{6}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t) - \frac{2}{3}e^{-4t}u(t) \tag{9}$$

Taking Laplace transforms:

right-sided

$$X(s) = \frac{1}{s+1} \qquad \mathcal{R}_x = \{\mathfrak{Re}\{s\} > -1\} \tag{10}$$

$$Y(s) = \frac{1/6}{s+1} + \frac{1/2}{s+3} - \frac{2/3}{s+4} \tag{11}$$



$$= \frac{(s+2)}{(s+1)(s+3)(s+4)} \qquad \mathcal{R}_y = \{\mathfrak{Re}\{s\} > -1\}. \tag{12}$$

do the algebra

Two-pole system: finding the impulse response

Dividing to get the transfer function:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{(s+2)}{(s+1)(s+3)(s+4)}}{\frac{1}{s+1}} \tag{13}$$

$$= \frac{(s+2)}{(s+3)(s+4)} \tag{14}$$

H should have 2 poles

PFE

$$= -\frac{1}{s+3} + \frac{2}{s+4} \qquad \mathcal{R}_h = \{\mathfrak{Re}\{s\} > -3\} \tag{15}$$

$$h(t) = -e^{-3t}u(t) + 2e^{-4t}u(t). \tag{16}$$

Example: noncausal system

Suppose we have the following input/output pair:

$$x(t) = e^{-3t}u(t) \tag{17}$$

$$y(t) = -\frac{1}{2}e^{-t}u(-t) - \frac{1}{2}e^{-3t}u(t). \tag{18}$$

left-sided

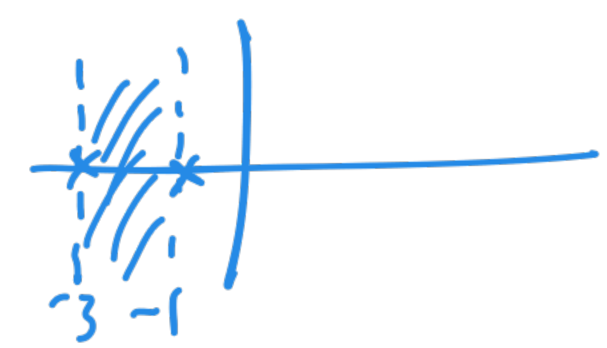
Here the input starts at 0 but the output extends from $-\infty < t < \infty$ – it's two sided. Taking Laplace transforms:

RoC Re{s} < -1

$$X(s) = \frac{1}{s+3} \qquad \mathcal{R}_x = \{\mathfrak{Re}\{s\} > -3\} \tag{19}$$

$$Y(s) = \frac{1/2}{s+1} + \frac{-1/2}{s+3} \tag{20}$$

RoC Re{s} > -3



$$= \frac{1}{(s+1)(s+3)} \qquad \mathcal{R}_y = \{-3 < \mathfrak{Re}\{s\} < -1\}. \tag{21}$$

Noncausal system: finding the impulse response

Dividing to get the transfer function:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{1}{(s+1)\cancel{(s+3)}}}{\frac{1}{\cancel{s+3}}} \tag{22}$$

$R_x \cap R_h = R_y$

$$= \frac{1}{s+1} \qquad \mathcal{R}_h = \{\mathfrak{Re}\{s\} < -1\} \tag{23}$$

to be consistent ROC has to be a LHP

$$h(t) = -e^{-t}u(-t). \tag{24}$$

Making sure the ROC is consistent

The trick is to make sure the regions of convergence work out. If you are given $\mathcal{R}_x$ and $\mathcal{R}_y$ then you need to choose an ROC $\mathcal{R}_h$ for $H(s)$ such that $\mathcal{R}_y = \mathcal{R}_x \cap \mathcal{R}_h$ (assuming no pole-zero cancellation). In our examples:

$$\mathcal{R}_x = \{\mathfrak{Re}\{s\} > -1\}, \mathcal{R}_y = \{\mathfrak{Re}\{s\} > -1\} \tag{25}$$
$$\implies \boxed{\mathcal{R}_h \supseteq \mathcal{R}_x} \tag{26}$$
$$\mathcal{R}_x = \{\mathfrak{Re}\{s\} > -3\}, \mathcal{R}_y = \{-3 < \mathfrak{Re}\{s\} < -1\} \tag{27}$$
$$\implies \mathcal{R}_h = \{\mathfrak{Re}\{s\} < -1\} \tag{28}$$

Drawing some pictures can help here.

Try it yourself

Problem

For these problems you can check yourself if you get the right answer. Given $x(t)$ *and* $h(t)$*, compute* $y(t)$ *using Laplace transforms. Then follow the same recipe we used earlier to re-compute* $h(t)$ *from* $x(t)$ *and* $y(t)$*. You can check that the* $h(t)$ *you got was the same as the* $h(t)$ *you started with.*

- $x(t) = e^{-4t}u(t)$, $h(t) = e^{-2t}u(t)$
- $x(t) = e^{-4t}u(t)$, $h(t) = 2e^{-3t}u(-t)$
- $x(t) = e^{-4t}u(t)$, $h(t) = e^{-2t}u(t) - 3e^{-6t}u(t)$