Linear Systems and Signals
An application: the inverted pendulum

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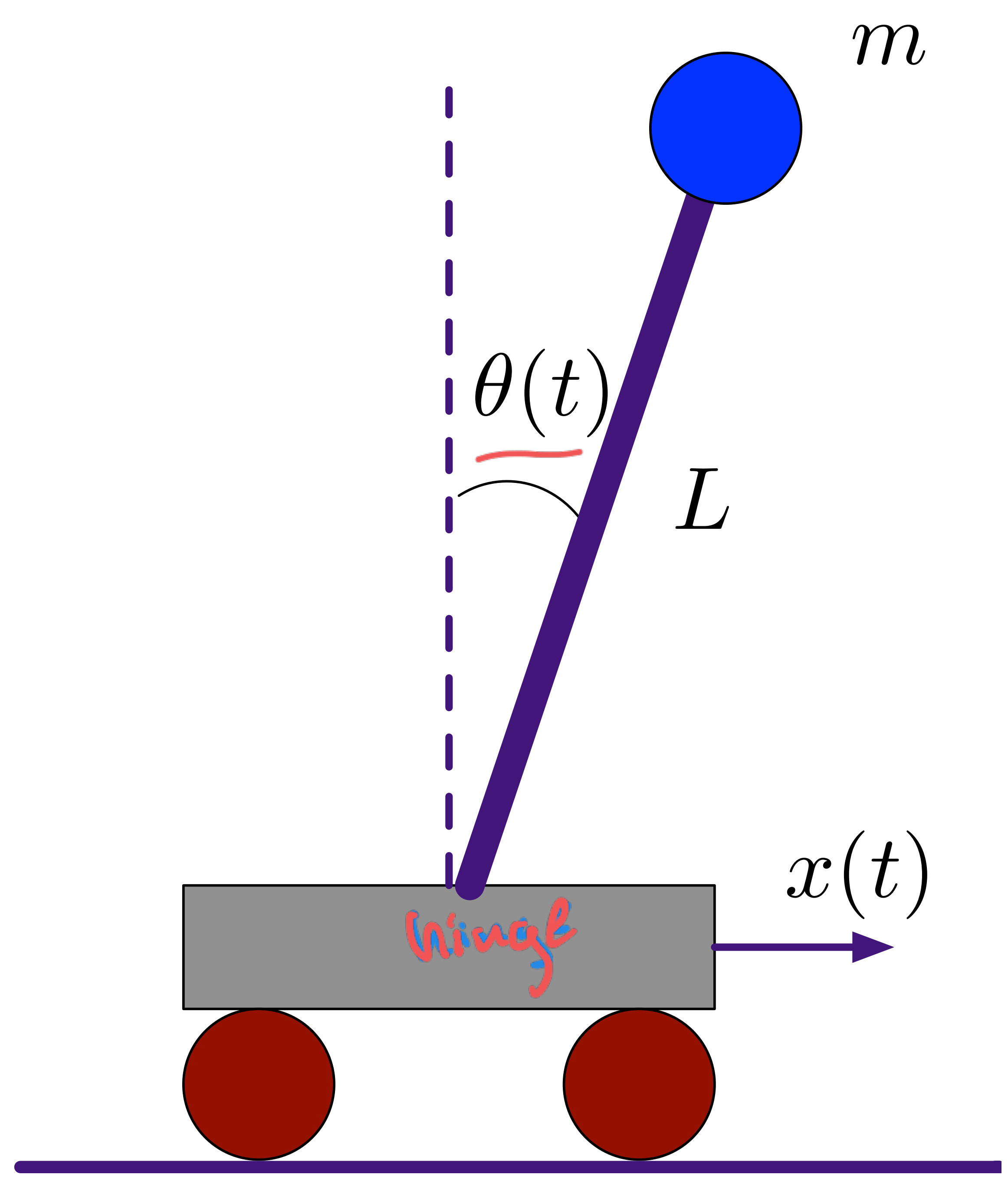
Learning objectives

The learning objectives for this section are:

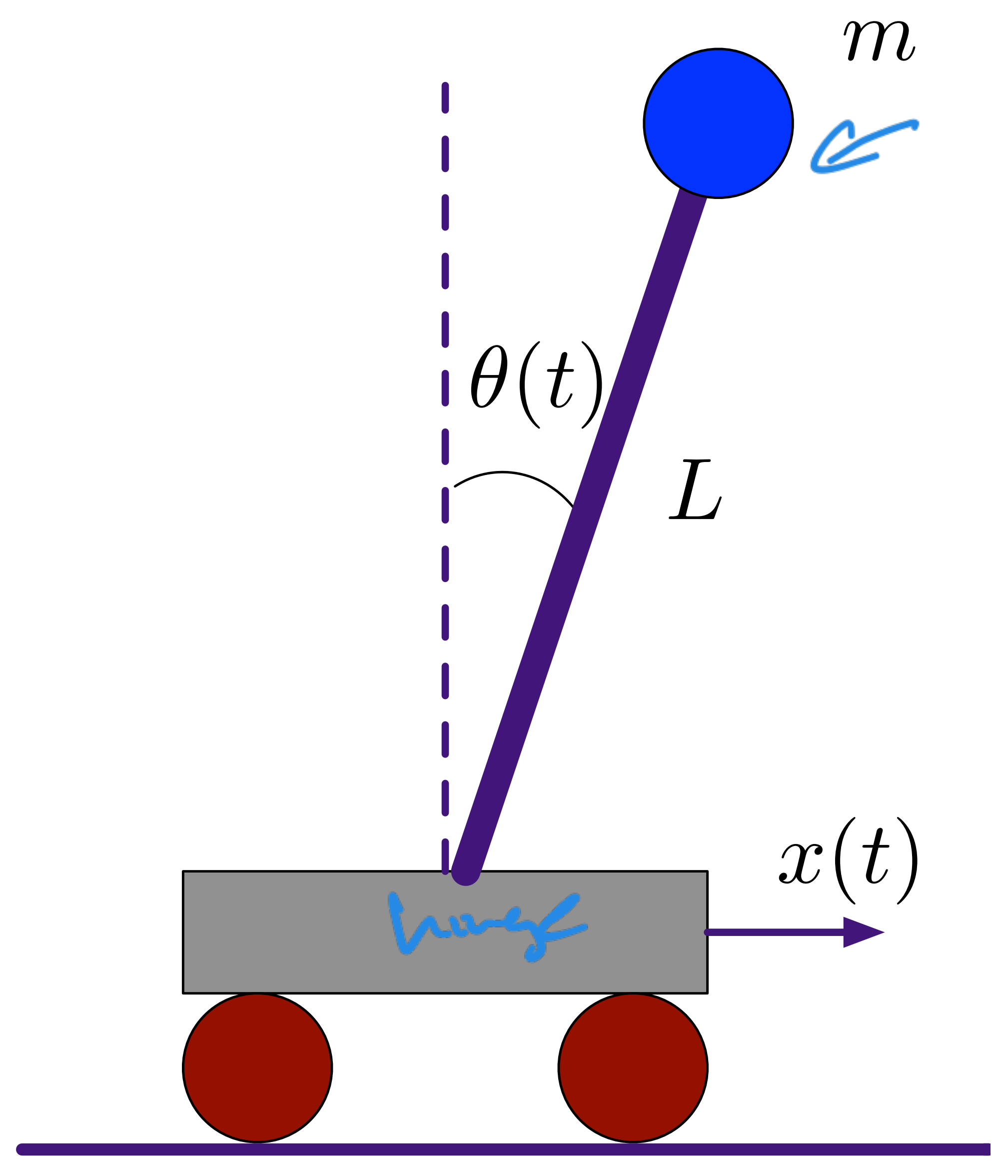
- understand how to model a physical system as an LTI system
- apply feedback control to stabilize a real-world system

The inverted pendulum



The inverted pendulum is a commonly-studied example for feedback control: a mass is balanced on a rod connected to a cart. We want to move the cart back and forth to keep the mass m vertical.

The physical set up



We'll simplify the model to assume the cart and rod are massless and the connector between the rod and cart is frictionless.

- $x(t)$ is the position of cart
- $\theta(t)$ is the angular displacement of the rod
- L is the length of the rod (important for torque)

Balancing torques

We have look at the different torques acting on the mass m.

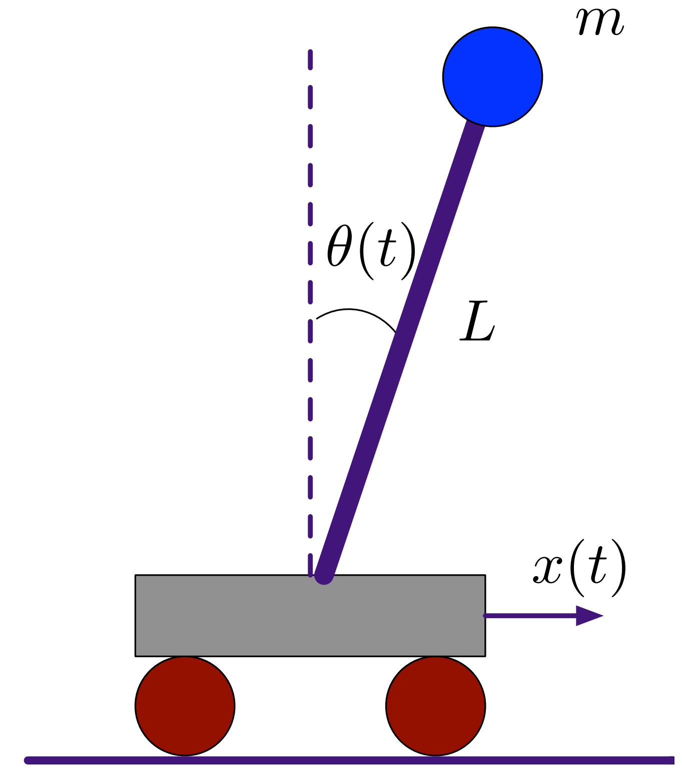
- $m(L\sin(\theta(t)))g$ is due to gravity on the mass, where $g = 9.8\frac{\mathrm{m}}{\mathrm{sec}^2}$.
- $-m(L\cos(\theta(t)))\frac{d^2}{dt^2}x(t)$ from the cart being displaced by $x(t)$.
- $mL^2\frac{d^2}{dt^2}\theta(t)$ is from angular acceleration (moment of inertia mL^2).

We have to set the angular acceleration torque equal to the sum of the two torques:

$$mL^2\frac{d^2}{dt^2}\theta(t) = m(L\sin(\theta(t)))g - m(L\cos(\theta(t)))\frac{d^2}{dt^2}x(t) \qquad (1)$$

This is *not* an LCCDE.

Approximation for small displacements



If θ is too big we have no hope of stabilizing the system, so let's assume $\theta(t) \ll 1$ remains small. In that case we can use a Taylor series to approximate the sine and cosine:

linearization of the functions system

$$\sin(\theta(t)) = \theta(t) - \frac{\theta(t)^3}{3!} + \cdots \approx \theta(t) \tag{2}$$

$$\cos(\theta(t)) = 1 - \frac{\theta(t)^2}{2!} + \cdots \approx 1 \tag{3}$$

Linearizing the differential equation

So this simplifies our differential equation to an LCCDE:

$$mL^2 \frac{d^2}{dt^2}\theta(t) = m(L\sin(\theta(t)))g - m(L\cos(\theta(t)))\frac{d^2}{dt^2}x(t) \tag{4}$$

$$mL^2 \frac{d^2}{dt^2}\theta(t) = mLg\theta(t) - mL\frac{d^2}{dt^2}x(t). \tag{5}$$

Now we can divide by mL^2:

$$\boxed{\frac{d^2}{dt^2}\theta(t) = \frac{g}{L}\theta(t) - \frac{1}{L}\frac{d^2}{dt^2}x(t).} \tag{6}$$

Taking Laplace transforms:

open loop plant

$$G(s) = \frac{\Theta(s)}{X(s)} = -\frac{s^2/L}{s^2 - g/L}. \tag{7}$$

Pole-zero analysis

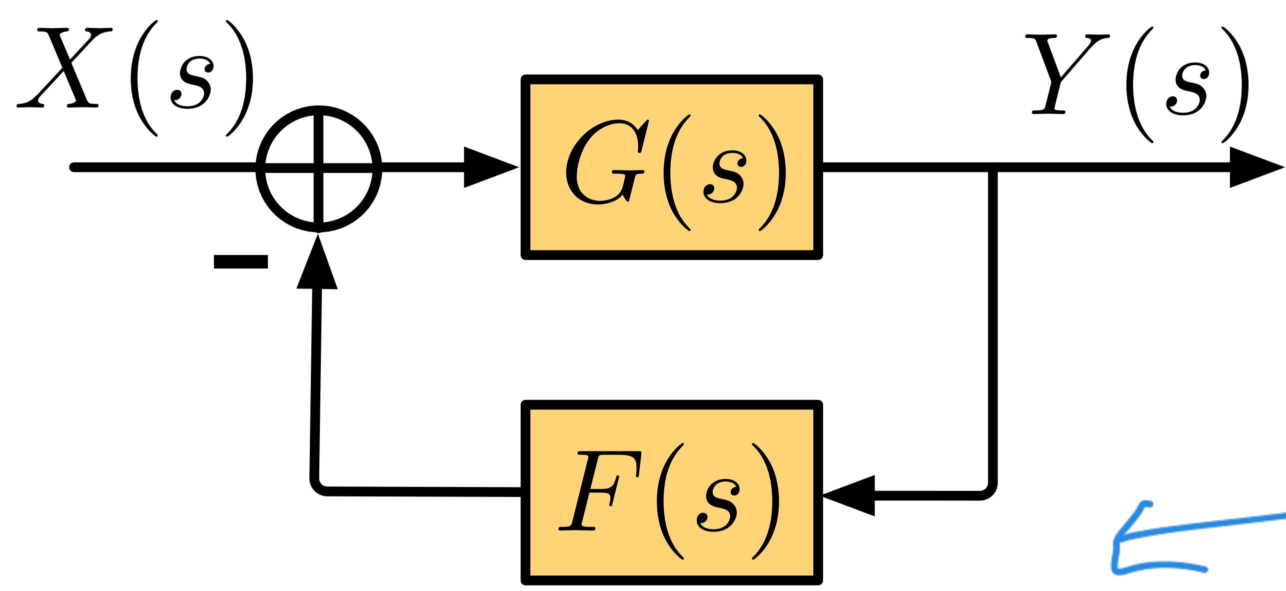
The transfer function is

$$G(s) = -\frac{s^2/L}{s^2 - g/L} = -\frac{s^2}{Ls^2 - g}. \tag{8}$$

Poles are at $\pm\sqrt{g/L}$. Since this physical system is causal, the ROC is $\Re\mathfrak{e}\{s\} > \sqrt{g/L}$ which does not contain the imaginary axis so the system is unstable.

- We can compute the output of the system to different signals $x(t)$ and see that the system output will blow up.
- Suppose we have a sensor that can measure $\theta(t)$. Can we use feedback to stabilize this system?

PI control



The plant is $G(s) = -\frac{s^2/L}{s^2 - g/L}$. Using closed-loop feedback control we can try to stabilize the system. Let's try proportional-integral (PI) control:

$$F(s) = K_p + K_i \frac{1}{s} \tag{9}$$

try to show P/PD control don't work

Then we have for the closed-loop system

$$H(s) = \frac{G(s)}{1 + F(s)G(s)} = \frac{-\frac{s^2}{Ls^2 - g}}{1 - (K_p + K_i/s)\frac{s^2}{Ls^2 - g}} \tag{10}$$

Finding the feedback system

The closed-loop system is

$$H(s) = \frac{-\frac{s^2}{Ls^2-g}}{1-(K_p+K_i/s)\frac{s^2}{Ls^2-g}} \tag{11}$$

$$= \frac{-s^2}{(L-K_p)s^2 - K_i s - g} \tag{12}$$

We have two zeros at 0 and two poles: with two parameters to control the system we can move the poles to many different locations. To get a stable system, let's first set $K_p = L + \alpha$, so

$$H(s) = \frac{s^2}{\alpha s^2 + K_i s + g} \tag{13}$$

Now we can find the pole locations and move them around.

Pole locations: root locus

Starting with

$$H(s) = \frac{s^2}{\alpha s^2 + K_i s + g} \qquad (14)$$

closed loop system

we have poles at

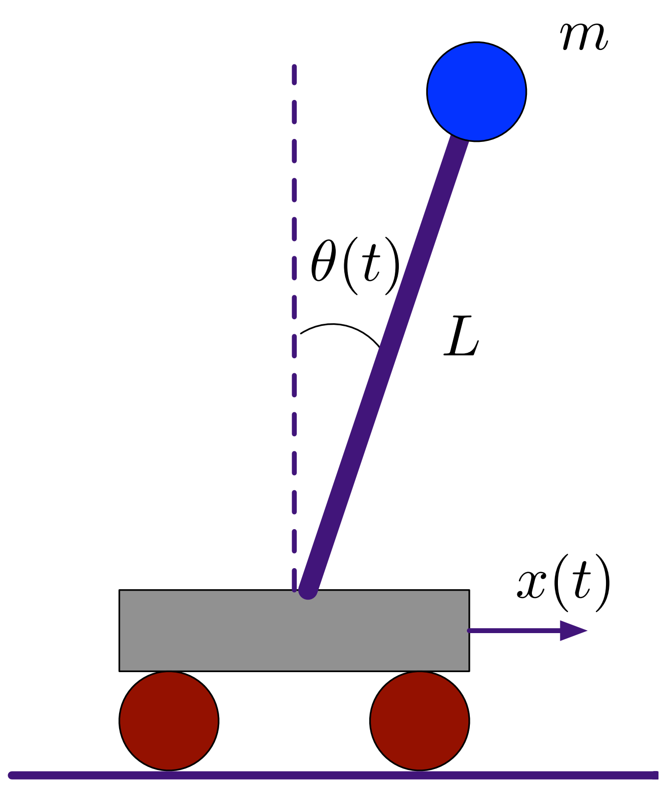
$$s = \frac{-K_i \pm \sqrt{K_i^2 - 4\alpha g}}{2\alpha}. \qquad (15)$$

$= 0$

root locus

We can see where these move numerically. We can consider the "*critically damped*" case where the poles are the same, in which case we have $K_i^2 = 4\alpha g$. For example, if $\alpha = 0.5$ then we can set $K_i = \sqrt{2g}$.

Recap



We started with a simplified physical system that we wanted to analyze.

- Use the physics to write down a differential equation describing the relationship between signals.
- Linearize the model by making appropriate assumptions about parameters/signal values.
- Generate the Laplace transform for the open-loop system.
- Close the loop by applying feedback control.
- Find parameters that can stabilize the closed-loop system.

More questions

We can write down a lot of different physical systems using RLC circuits or spring-mass-damper mechanical systems. These will lead to Laplace transforms that we can then analyze and apply feedback for stabilization.

Problem

Try proportional (P) and proportional-derivative (PD) control on the inverted pendulum and convince yourself that these policies cannot stabilize the system.
Try a PID control system and see how this might change how you can move the poles and zeros around. Does the added derivative control help that much?

$$F(s) = K_p + \underline{K_d s} + \underline{K_i \frac{1}{s}}$$