Linear Systems and Signals

System properties, the ROC, and inverse systems

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Learning objectives

The learning objectives for this section are:

- characterize stability and causality of a system in terms of the ROC
- find all time domain signals with a given transform
- determine if causal, stable inverses exist

System properties

We can relate system properties to the ROC of a rational transform $H(s)$ with ROC $\mathcal{R}_h$:

don't allow delay $e^{-s\tau}$

- If $H(s)$ is causal system then $\mathcal{R}_h$ goes from the rightmost pole to ∞.

  h(t) is right-sided

- If $H(s)$ is anticausal system then $\mathcal{R}_h$ goes from $-\infty$ to the leftmost pole.
- $H(s)$ is *stable* if and only if $\mathcal{R}_h$ contains the imaginary axis $\{s = j\omega\}$.

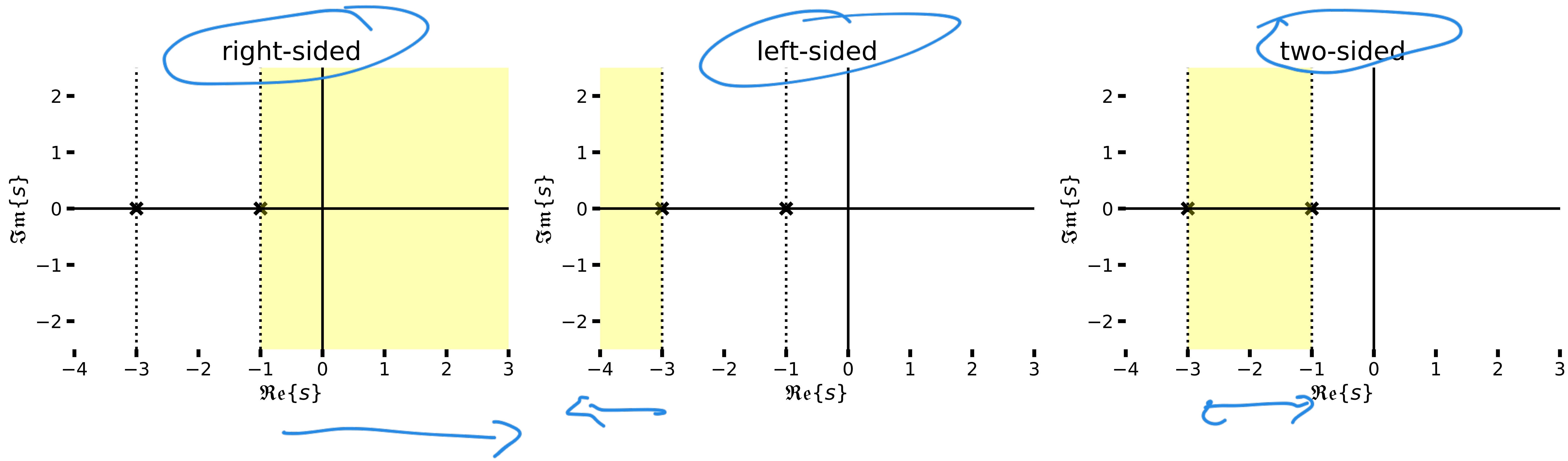
  causal + stable $\Rightarrow$ all poles are in the left half-plane.

  $\frac{1}{H(s)}$

- $H(s)$ is *invertible* if there is a $G(s)$ with ROC $\mathcal{R}_g$ such that $H(s)G(s) = 1$ and $\mathcal{R}_h \cap \mathcal{R}_g \neq \emptyset$

We can use these properties to find a time domain signal with a given Laplace transform.

Different signals, same transform, different ROCs



Recall that multiple signals can have the same Laplace transform with different ROCs. For example:

zero @ 1
poles @ −3, −1

$$Y(s) = \frac{(s-1)}{(s+3)(s+1)} = \frac{2}{s+3} - \frac{1}{s+1} \tag{1}$$

can come from

right
$$y_1(t) = 2e^{-3t}u(t) - e^{-t}u(t) \tag{2}$$

left
$$y_2(t) = -2e^{-3t}u(-t) + e^{-t}u(-t) \tag{3}$$

two
$$y_3(t) = 2e^{-3t}u(t) + e^{-t}u(-t) \tag{4}$$

Convolution and the ROCs

The convolution property says that if $h(t) \overset{\mathcal{L}}{\longleftrightarrow} H(s)$ and $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ then

$$y(t) = (h * x)(t) \overset{\mathcal{L}}{\longleftrightarrow} Y(s) = H(s)X(s) \tag{5}$$

where the ROC contains $\mathcal{R}_x \cap \mathcal{R}_h$. Suppose we have

$$X(s) = \frac{s-1}{s+1} = 1 - \frac{2}{s+1} \tag{6}$$

$$H(s) = \frac{1}{s+3} \tag{7}$$

$$Y(s) = \frac{(s-1)}{(s+3)(s+1)} \tag{8}$$

Four options?

We have two different $x(t)$ signals with the same $X(s)$ but different $\mathcal{R}_x$. Likewise, we have two different $h(t)$ signals with the same $H(s)$.

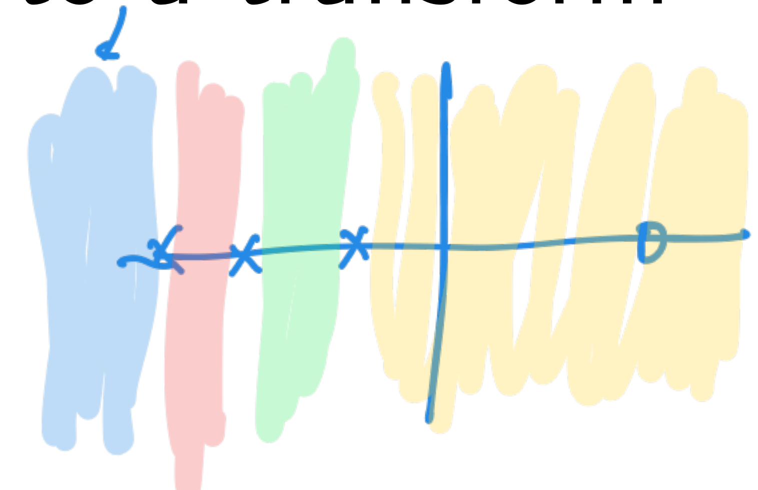
$h(t)$ \ $x(t)$	right $\delta(t) - 2e^{-t}u(t)$	left $\delta(t) + 2e^{-t}u(-t)$
right $e^{-3t}u(t)$	right-sided $y_1(t)$	two-sided $y_3(t)$
left $-e^{-3t}u(-t)$	no intersection	left-sided $y_2(t)$

In order for $y(t)$ to exist the intersection $\mathcal{R}_x \cap \mathcal{R}_h$ cannot be empty.

Example: finding the ROCs

Let's find all the impulse responses which can lead to a transform

$$H(s) = \frac{s+5}{(s+2)(s+4)(s+6)} \tag{9}$$



This has a zero at $s = -5$ and poles at $s = -2$, $s = -6$, and $s = -4$.
Possible ROCs:

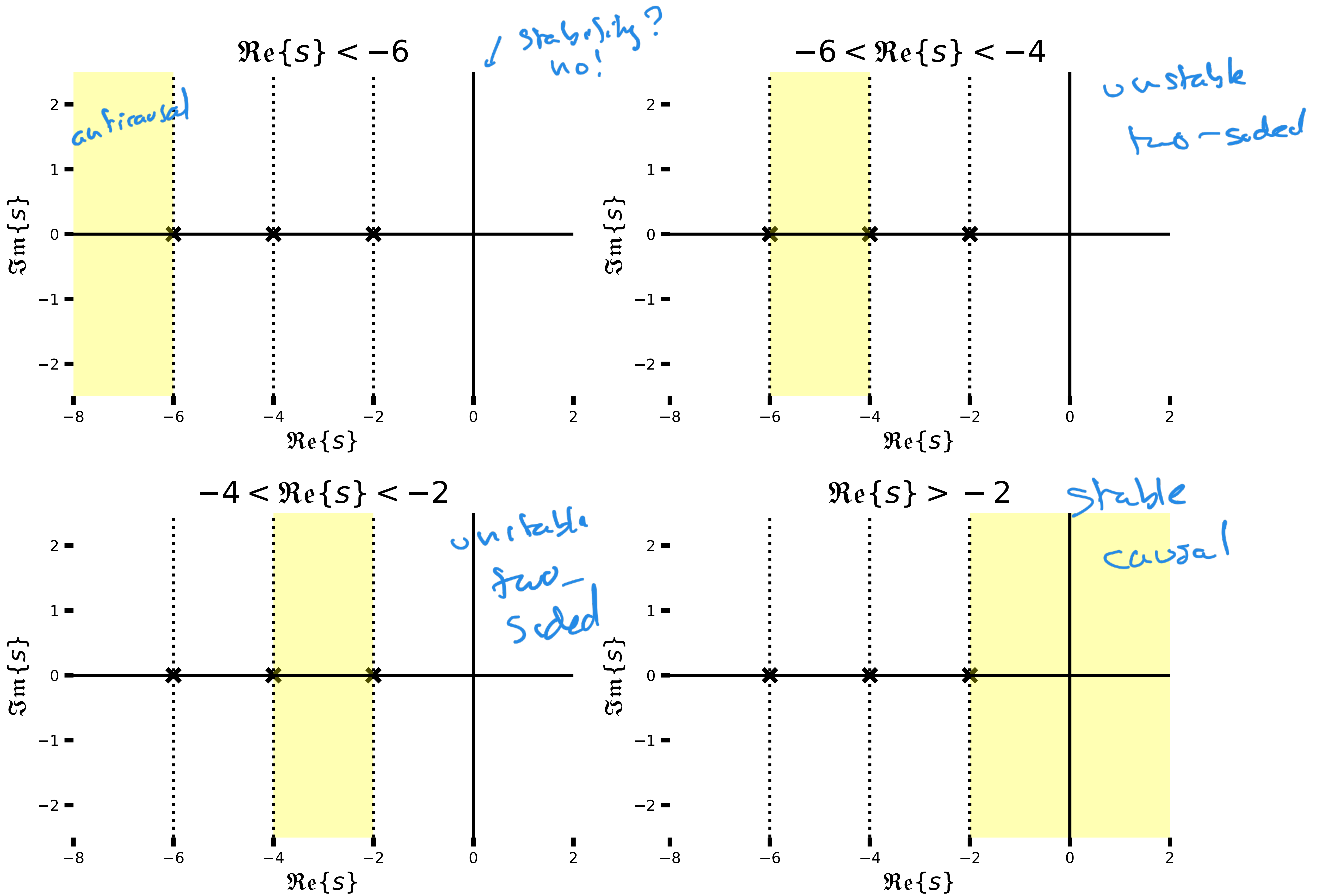
$$\{\Re\{s\} < -6\} \qquad \text{left-sided, unstable} \tag{10}$$

$$\{-6 < \Re\{s\} < -4\} \qquad \text{two-sided, unstable} \tag{11}$$

$$\{-4 < \Re\{s\} < -2\} \qquad \text{two-sided, unstable} \tag{12}$$

$$\{\Re\{s\} > -2\} \qquad \text{right-sided, stable} \tag{13}$$

The pole-zero diagram



Split using partial fraction expansion

Using partial fraction expansion:

$$\left.\frac{s+5}{(s+4)(s+6)}\right|_{s=-2} = \frac{3}{8} \tag{14}$$

$$\left.\frac{s+5}{(s+2)(s+6)}\right|_{s=-4} = -\frac{1}{4} \tag{15}$$

$$\left.\frac{s+5}{(s+2)(s+4)}\right|_{s=-6} = -\frac{1}{8} \tag{16}$$

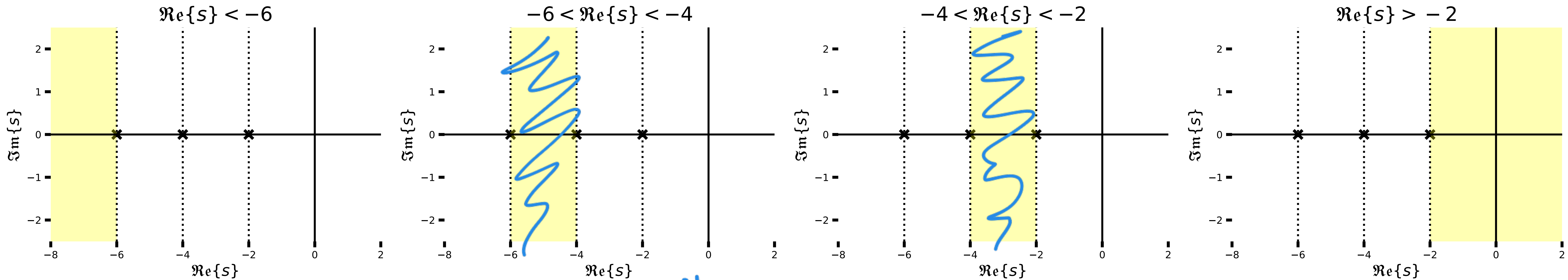
So

R_1 R_2 R_3

$$\boxed{H(s) = \frac{3/8}{s+2} - \frac{1/4}{s+4} - \frac{1/8}{s+6}.} \tag{17}$$

So for each of the corresponding ROCs we have to choose which inverse (left- or right-sided) time-domain signal to choose.

Associating the time-domain signals



$-e^{-2t}u(-t)$

$-e^{-4t}u(-t)$

$e^{-6t}u(t)$

left sided

$$h(t) = -\frac{3}{8}e^{-2t}u(-t) + \frac{1}{4}e^{-4t}u(-t) + \frac{1}{8}e^{-6t}u(-t) \qquad \{\Re\{s\} < -6\}$$

$$h(t) = -\frac{3}{8}e^{-2t}u(-t) + \frac{1}{4}e^{-4t}u(-t) - \frac{1}{8}e^{-6t}u(t) \qquad \{-6 < \Re\{s\} < -4\}$$

$$h(t) = -\frac{3}{8}e^{-2t}u(-t) - \frac{1}{4}e^{-4t}u(t) - \frac{1}{8}e^{-6t}u(t) \qquad \{-4 < \Re\{s\} < -2\}$$

right-sided

$$h(t) = \frac{3}{8}e^{-2t}u(t) - \frac{1}{4}e^{-4t}u(t) - \frac{1}{8}e^{-6t}u(t) \qquad \{\Re\{s\} > -2\}$$

Determining inverses

Does the system with the following impulse response have a stable, causal inverse?

$$H(s) = \frac{(s+1)(s+2)}{s+3} \qquad \mathcal{R}_h = \{\mathfrak{Re}\{s\} > -3\} \tag{18}$$

Let's take the algebraic inverse:

PFE

$$\frac{1}{H(s)} = G(s) = \frac{(s+3)}{(s+1)(s+2)} = \boxed{\frac{2}{s+1} - \frac{1}{s+2}} \tag{19}$$

This has 3 ROCs: $\{\mathfrak{Re}\{s\} < -2\}$, $\{-2 < \mathfrak{Re}\{s\} < -1\}$, and $\{\mathfrak{Re}\{s\} > -1\}$. The last one is stable and causal, and

contains imag axis $\Rightarrow$ stable

$$\text{[redacted]} \{\mathfrak{Re}\{s\} > -3\} \cap \{\mathfrak{Re}\{s\} > -1\} = \boxed{\{\mathfrak{Re}\{s\} > -1\}} \tag{20}$$

So $g(t) = 2e^{-t}u(t) - e^{-2t}u(t)$ is a stable causal inverse.

Sometimes inverses are not realizable

Does the system with the following impulse response have a stable, and causal inverse?

$$H(s) = \frac{(s-3)}{(s+1)(s+2)} \qquad \boxed{\mathcal{R}_h = \{\mathfrak{Re}\{s\} > -1\}} \tag{21}$$

Taking the algebraic inverse:

$= \frac{1}{H(s)}$ long division

$$G(s) = \frac{(s+1)(s+2)}{(s-3)} = \frac{s^2+3s+2}{s-3} = \boxed{s + 6 - \frac{16}{s-3}}. \tag{22}$$

This has 2 ROCs: $\{\mathfrak{Re}\{s\} < 3\}$, $\{\mathfrak{Re}\{s\} > 3\}$. We have two choices:

$\{-1 < \mathfrak{Re}\{s\} < 3\}$ stable, two-sided

$\{\mathfrak{Re}\{s\} > 3\}$ unstable, right-sided

So the answer is no.

Try it yourself

Problem

Find all time domain signals with the following Laplace transforms and determine if the systems are causal/anticausal or stable/unstable:

$$H(s) = \frac{s-3}{s^2+5s+6} \tag{23}$$

$$H(s) = \frac{s-1}{2s^2+11s+12} \tag{24}$$

$$H(s) = \frac{s+6}{s^2+2s-15} \tag{25}$$

$$H(s) = \frac{s+2}{s^3+3s^2-s-3} \tag{26}$$

Determine if there are causal, stable inverses for each of them.