Linear Systems and Signals
Eigenfunctions of CT LTI systems

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Learning objectives

The learning objectives for this section are:

- use the eigenfunction property to compute the output of LTI systems with complex exponential input
- show that LTI systems cannot add new frequencies to the input function

Eigenvectors and eigenfunctions

Remember that for DT systems we saw that *complex exponentials are eigenfunctions of LTI systems*. The same property holds for CT LTI systems. Suppose $x(t) = e^{j\omega_0 t}$. Then

mag 1 phase $= \omega_0 t$

convolution

$$\int_{-\infty}^{\infty} h(\tau) e^{j\omega_0 (t-\tau)} d\tau = e^{j\omega_0 t} \boxed{\int_{-\infty}^{\infty} h(\tau) e^{-j\omega_0 \tau} d\tau} \qquad (1)$$

just a function of h and ω_0

If we define

$$\boxed{H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau} \qquad (2)$$

CTFT

we get

$$h(t) * e^{j\omega_0 t} = H(j\omega_0) e^{j\omega_0 t}. \qquad (3)$$

$H(j\omega)$ is the *Continuous Time Fourier Transform* (more on that later).

How a system acts on a complex exponential

$$e^{j\omega_0 t} \longrightarrow \boxed{h[n]} \longrightarrow H(j\omega_0)e^{j\omega_0 t}$$

The quantity

$$\boxed{H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt} \quad (4)$$

is a complex-valued function, so putting it in magnitude-phase form $|H(j\omega)|\cancel{e^{\measuredangle H(j\omega)}}$ $e^{j\measuredangle H(j\omega)}$, for $x(t) = e^{j\omega_0 t}$ we get

$$(h * x)(t) = |H(j\omega_0)|e^{j(\omega_0 t + \measuredangle H(j\omega_0))} \quad (5)$$

That means the LTI system changes the *magnitude* and *phase* of a complex exponential but not the frequency.

Implications

$$e^{j\omega_0 t} \longrightarrow \boxed{h[n]} \longrightarrow H(j\omega_0)e^{j\omega_0 t}$$

- LTI systems multiply complex exponentials signals by a (complex-valued) constant. This is why we say *complex exponentials are eigenfunctions of LTI systems*.
- LTI systems cannot create new frequencies not present in the original signal.
- Linear combinations of complex exponentials (example: cosines) produce linear combinations of complex exponentials at the same frequencies.

Symmetry

Suppose $h(t)$ is a real-valued signal. Take the complex conjugate of H:

$$H^*(j\omega) = \left(\int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \right)^* \tag{6}$$

$$= \int_{-\infty}^{\infty} h^*(t) e^{j\omega t} dt \qquad h^*[t] = h(t) \tag{7}$$

$$= H(-j\omega) \tag{8}$$

So as a function of ω, $H(j\omega)$ is *conjuate symmetric*. This implies that

$$|H(j\omega)| \text{ is even} \qquad \measuredangle H(j\omega) \text{ is odd} \tag{9}$$

Prove that the DT version $H(e^{j\omega})$ is also conjugate symmetric with even magnitude and odd phase.

An example

Suppose $h(t) = e^{-3t}u(t)$. Then

$$H(j\omega) = \int_{-\infty}^{\infty} e^{-3t} e^{-j\omega t} u(t) dt \tag{10}$$

$$= \int_{0}^{\infty} e^{-(3+j\omega)t} dt \tag{11}$$

$$= \frac{1}{3+j\omega} \tag{12}$$

$$= \frac{3}{9+\omega^2} - j\frac{\omega}{9+\omega^2} \tag{13}$$

$$= \frac{1}{9+\omega^2} e^{-j\tan^{-1}(\omega/3)}. \tag{14}$$

$$\frac{3^2 + \omega^2}{(9+\omega^2)^2} = \frac{1}{9+\omega^2}$$

cartesian

mag phase

This means an input $e^{j\omega_0 t}$ gets a gain of $\frac{1}{9+\omega_0^2}$ and a phase shift of $-\tan^{-1}(\omega_0/3)$.

Example continued

Suppose $h(t) = e^{-3t}u(t)$ and $x(t) = \cos(6\pi t)$. Eulerizing $x(t)$:

$$x(t) = \frac{1}{2}e^{j6\pi t} + \frac{1}{2}e^{-j6\pi t} \tag{15}$$

So now the output $y = (x * h)(t)$, we have (since $\omega_0 = +6\pi$ for the first term and -6π for the second term) :

$\frac{\omega_0}{3} = \frac{-6\pi}{3}$

$$y(t) = \frac{1}{2}\frac{1}{9+36\pi^2}e^{j6\pi t - j\tan^{-1}\frac{1}{2\pi}} + \frac{1}{2}\frac{1}{9+36\pi^2}e^{-j6\pi t - j\tan^{-1}\frac{-1}{2\pi}} \tag{16}$$

$$y(t) = \frac{1}{9+36\pi^2}\left(\frac{1}{2}e^{j(6\pi t - \tan^{-1}\frac{1}{2\pi})} + \frac{1}{2}e^{-j(6\pi t - \tan^{-1}\frac{1}{2\pi})}\right) \tag{17}$$

$$= \frac{1}{9+36\pi^2}\cos\left(6\pi t - \tan^{-1}\frac{1}{2\pi}\right) \tag{18}$$

The general case

For general $h(t)$ and $x(t) = \cos(\omega t)$ we get

$$y(t) = H(j\omega)\left(\frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t}\right) \tag{19}$$

Eulerize

mag even
phase odd

$$= |H(j\omega)|\frac{1}{2}e^{j(\omega t + \measuredangle H(j\omega))} + |H(-j\omega)|\frac{1}{2}e^{-j(\omega t + \measuredangle H(-j\omega))} \tag{20}$$

$$= |H(j\omega)|\frac{1}{2}e^{j(\omega t + \measuredangle H(j\omega))} + |H(j\omega)|\frac{1}{2}e^{-j(\omega t - \measuredangle H(j\omega))} \tag{21}$$

$$= |H(j\omega)|\cos(\omega t + \measuredangle H(j\omega)). \tag{22}$$

This is called the sinusoidal response of the system.

Check for yourself: what is the formula for an input of $\sin(\omega t)$?

Try it yourself

Problem

Find the sinusoidal responses for these combinations of input and impulse response:

$$x(t) = \cos(100\pi n), h(t) = e^{-4t+2}u(t) \tag{23}$$

$$x(t) = \cos(4\pi t), h(t) = \delta(t-3) \tag{24}$$

$$x(t) = 4\sin(8\pi t + \pi/3), h(t) = e^{-2t}u(t) \tag{25}$$

Make up some on your own!