Linear Systems and Signals
Convolution properties

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Learning objectives

The learning objective for this section is:

- ~~apply~~ understand properties of the convolution formula

Many cheerful facts about the convolution

We write $y[n] = (x * h)[n]$ or $y = x * h$ for

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = (x*h)[n]$$

The convolution has some nice properties:

- *Commutative*: $\boxed{(x * h) = (h * x)}$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

- *Associative*: $(x * h) * g = x * (h * g)$.
- *Distributive*: $x * (g + h) = (x * g) + (x * h)$.

Commutativity: $(x * h) = (h * x)$

1. Start with the definition:

n-m

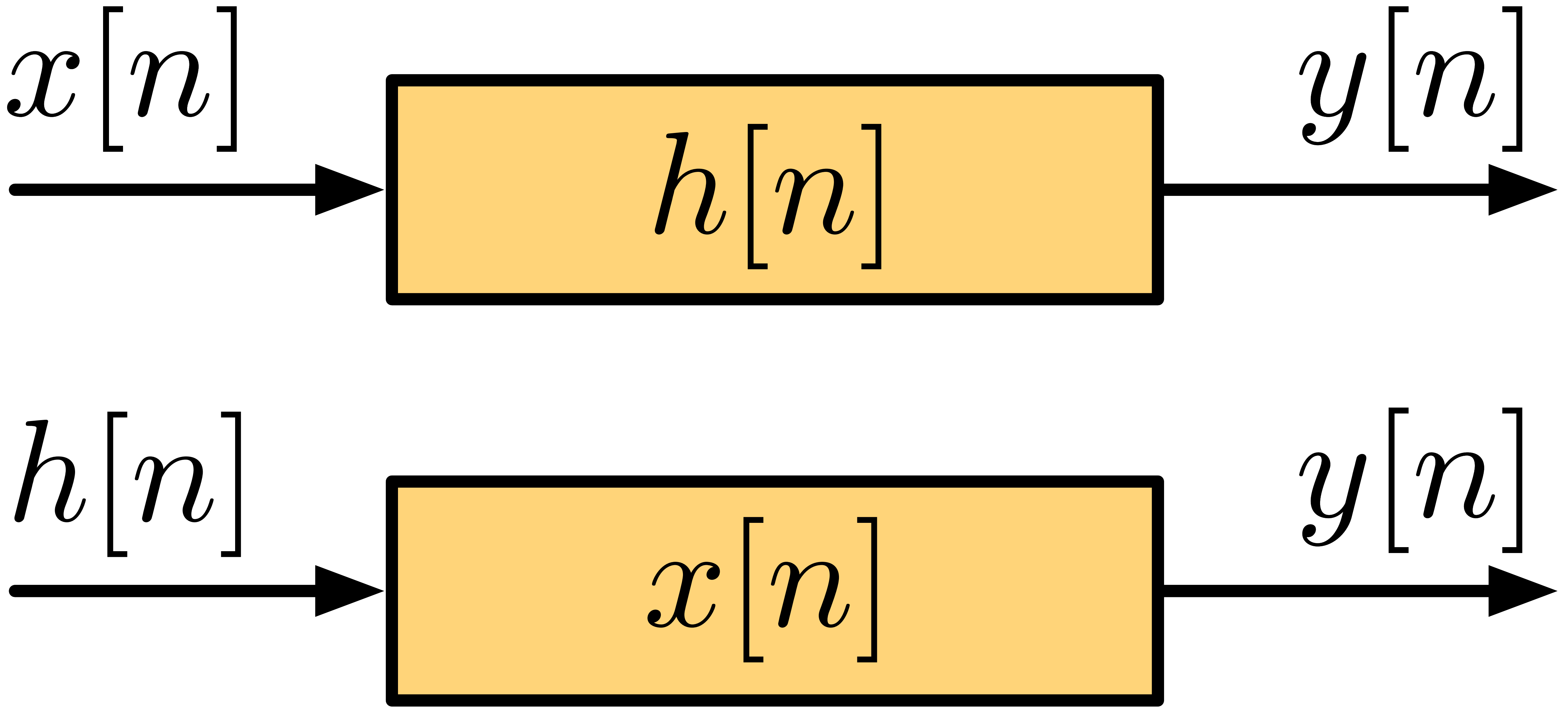
= m

$$(h * x)[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

2. Set $m = n - k$ so $k = n - m$. For any fixed n, as $k = -\infty$ to $+\infty$, m goes from $+\infty$ to $-\infty$:

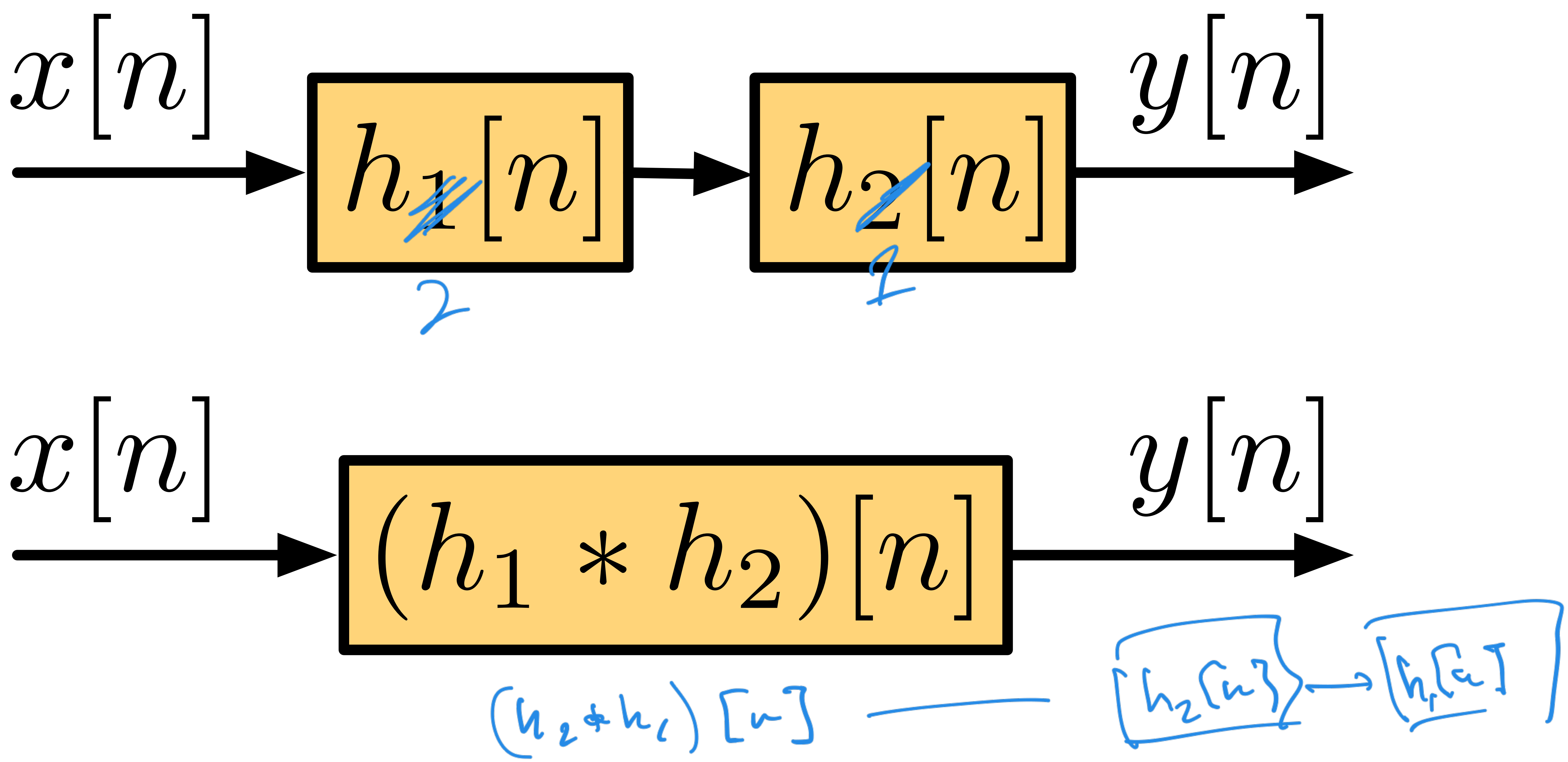
$$\begin{aligned}(h * x)[n] &= \sum_{m=-\infty}^{\infty} h[n-m]x[m] \\ &= \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\ &= (x * h)[n]\end{aligned}$$

Why is commutativity useful?



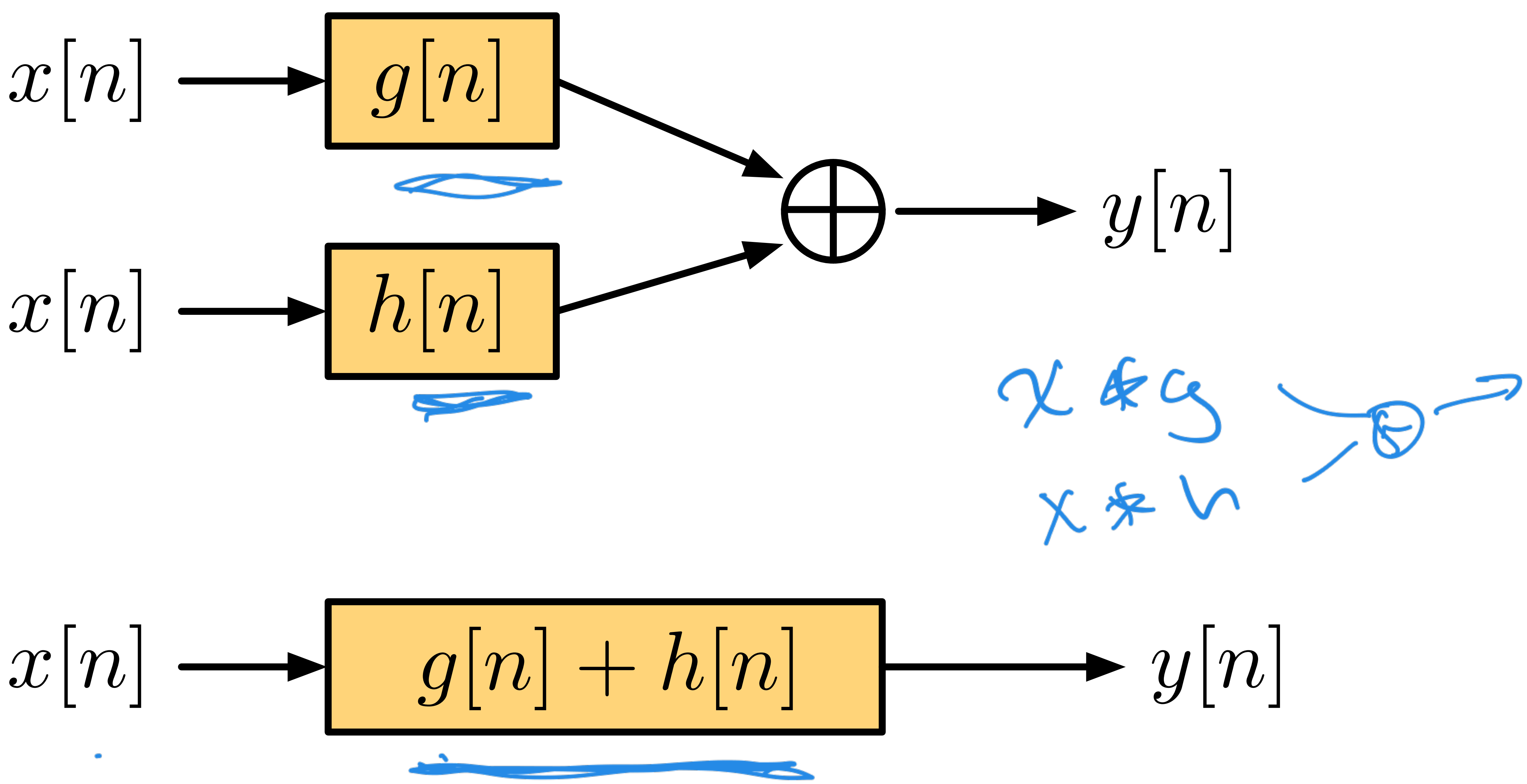
- Sometimes the calculation is easier one way than another.
- Can also think of $h[n]$ as a signal: lets us manipulate block diagrams.

Why is associativity useful?



- Merge/split cascades of systems
- With commutativity, swap the order of systems

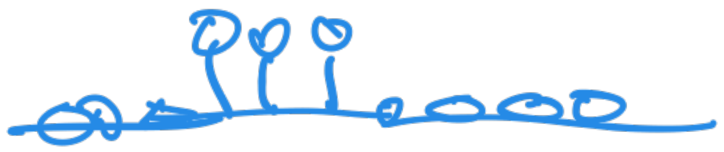
Why is distributivity useful?



- Understand what systems are doing by looking at what each part does separately.
- Simplify block diagrams to get overall input/output behavior

Some language around DT LTI systems

- The **impulse response** is the output of the system when the input is $\delta[n]$, the unit impulse function.
- We often call an LTI system or its impulse ~~a~~ response a **filter**.
- If $h[n]$ is only $\neq 0$ for a finite number of time points, we call it an **finite impulse response** (FIR) filter.



FIR

- If $h[n]$ is $\neq 0$ for an infinite number of time points, we call it an **infinite impulse response** (IIR) filter.

$h[n] = \alpha^n u[n]$

IIR

The impulse response is everything

Critical fact: The output of a DT LTI system is only a function of the **input signal** and **impulse response**.

- We can find all of the system properties from the impulse response: causality, stability, etc.
- By relating the values in the impulse response relates to the what the system *does*, we can start to design systems for specific tasks.