Linear Systems and Signals
Reusing general formulas

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Learning objectives

The learning objectives for this section are:

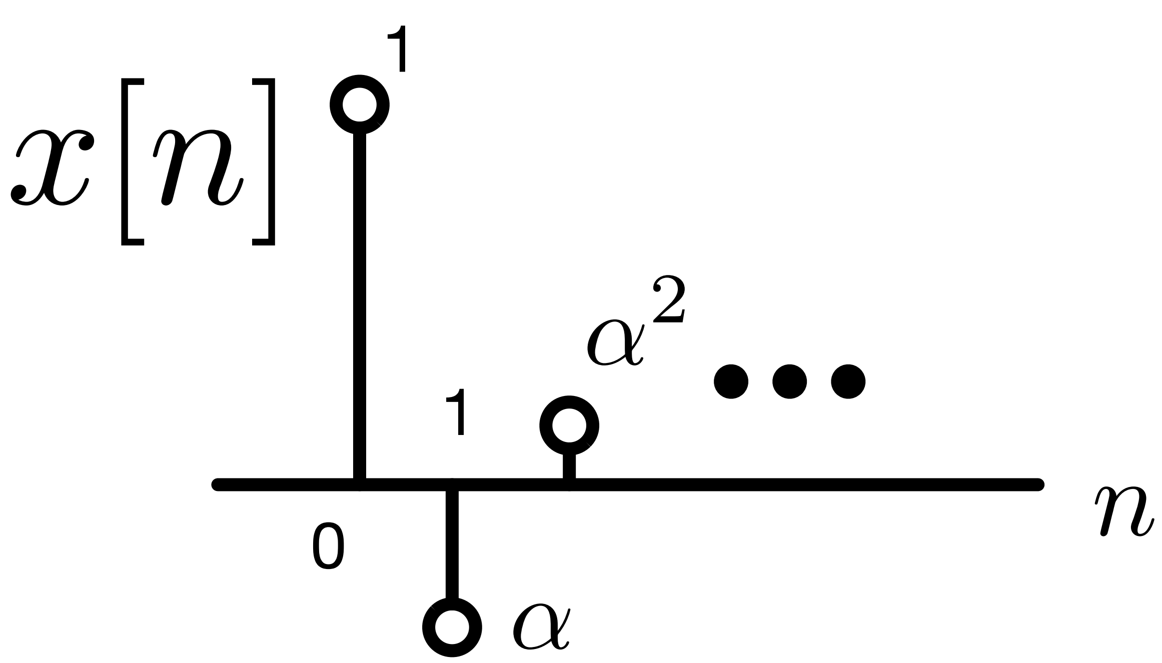
- use standard formulas to simplify output calculations
- use block diagrams to simplify convolutions

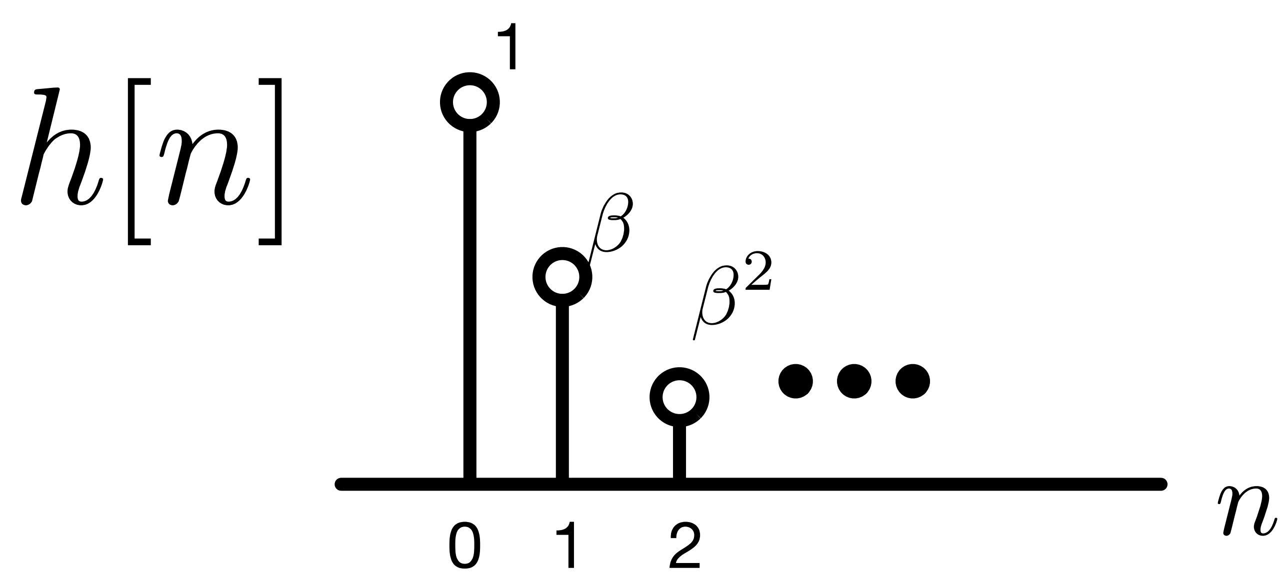
A general formula for exponentials

Find $y[n] = (x * h)[n]$ when $|\alpha| < |\beta|$ and

$$x[n] = \alpha^n u[n] \qquad |\alpha| < 1$$
$$h[n] = \beta^n u[n] \qquad |\beta| < 1$$

Step 0: draw a picture and rewrite the signals if needed:





$\alpha < 0$

$\beta > 0$

Writing out the convolution explicitly

1. Substitute $x[k]$ and $h[k]$ into the convolution formula. Use the step function to simplify the limits

$$y[n] = \sum_{k=-\infty}^{\infty} \alpha^k \beta^{n-k} u[k] u[n-k]$$

$$= \sum_{k=0}^{n} \alpha^k \beta^{n-k} u[n]$$

$k \geq 0$
$k \leq n$

2. Do some algebra to let us apply series formulas. Since $|\alpha| < |\beta|$

$$y[n] = \beta^n \sum_{m=0}^{n} \left(\frac{\alpha}{\beta}\right)^k u[n]$$

geometric series

$$= \beta^n \frac{1-(\alpha/\beta)^{n+1}}{1-\alpha/\beta} u[n] = \boxed{\frac{\beta^{n+1}-\alpha^{n+1}}{\beta-\alpha} u[n]}$$

How can we use this?

This formula is useful when we are faced with convolutions which look almost like this:

$$y[n] = x[n] * h[n] = \left(\left(\frac{1}{2}\right)^{n+2} u[n-1] \right) * \left(\left(\frac{1}{5}\right)^{n-1} u[n-4] \right)$$

decaying exp

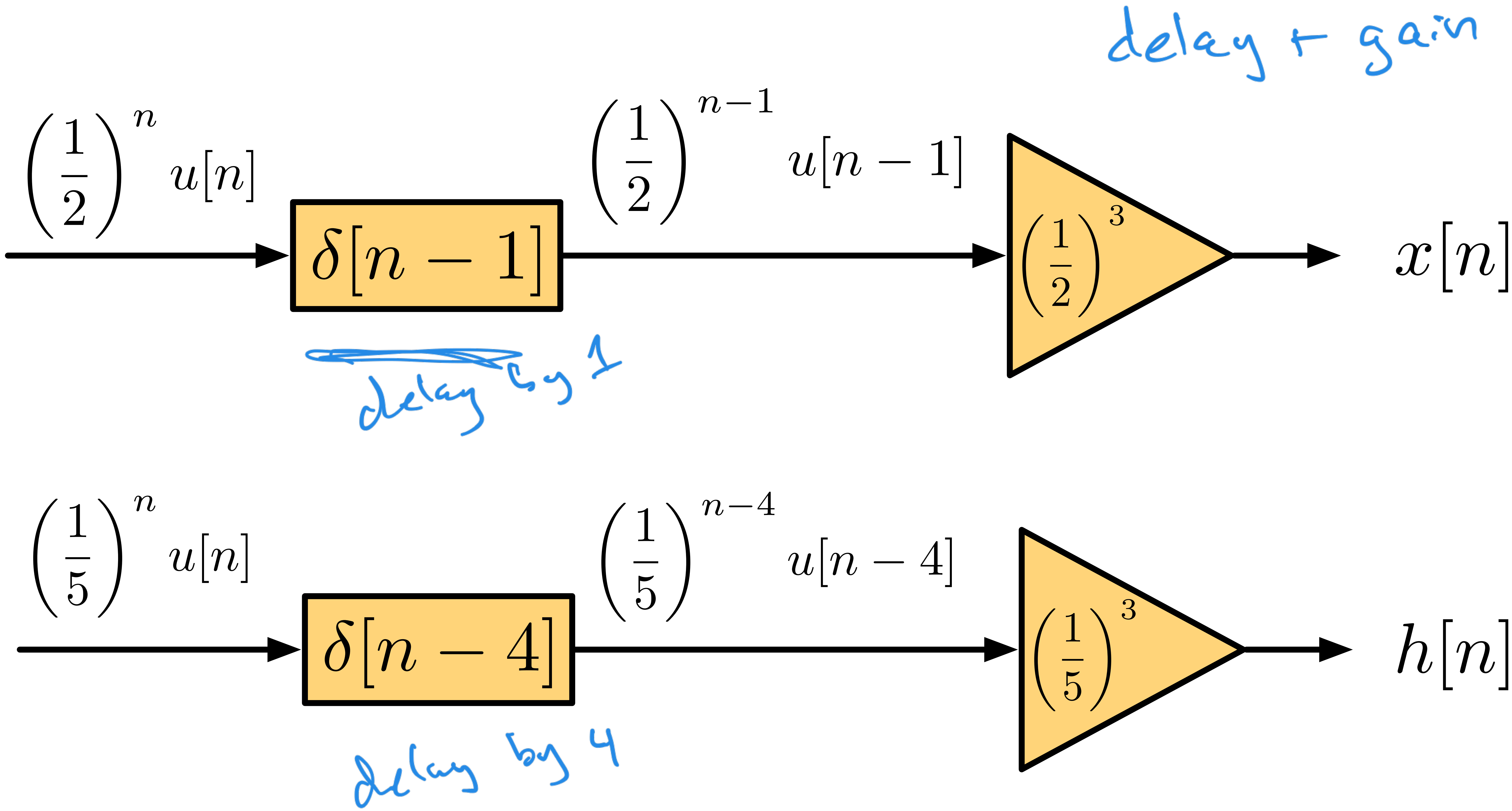
delay

How do we deal with this?

Step 0: draw a picture! This time, draw a picture of how you get these signals from signals you know about.

$\alpha^n u[n]$

Using block diagrams to get the signal



$$\alpha = \frac{1}{5} < \frac{1}{2} = \beta$$

Using block diagrams to get the signal

To combine these, let

we know $(a*b)[n]$

$$a[n] = \left(\frac{1}{5}\right)^n u[n] \qquad b[n] = \left(\frac{1}{2}\right)^n u[n] \tag{1}$$

Then

block diagram + commutativity of convolution

$$x[n] = b[n] * \delta[n-1] * \frac{1}{2^3}\delta[n] \tag{2}$$

$*$

$$h[n] = a[n] * \delta[n-4] * \frac{1}{5^3}\delta[n] \tag{3}$$

Use commutativity to get

delay 4+1 total

gather terms

$$(x*h)[n] = \boxed{\frac{(1/2)^{n+1} - (1/5)^{n+1}}{1/2 - 1/5} u[n]} * \delta[n-5] * \frac{1}{10^3}\delta[n] \tag{4}$$

template

total gain $\frac{1}{(2\cdot 5)^3}$

$$= \frac{1}{400}\left(\frac{1}{2}\right)^{n-4} u[n-5] - \frac{1}{400}\left(\frac{1}{5}\right)^{n-4} u[n-5] \tag{5}$$

General recipe

If you know $(a * b)[n]$ for some $a[n]$ and $b[n]$,

- Write $x[n]$ and $h[n]$ in terms of $a[n]$ and $b[n]$ passed through LTI filters (gain, delay, etc.).
- The convolution $(x * h)[n]$ is the convolution of all of these terms. Use commutativity to simplify/merge terms.
- Write the output as $(a * b)[n]$ passed through the merged LTI filters.

total gain
total delay

Try some yourself

Problem

Find the convolution from the following input-output relations:

$$h[n] = (1/3)^n u[n], x[n] = (1/6)^n u[n]$$
$$h[n] = (1/4)^n u[n], (-1/2)^{n-1} u[n+3]$$
$$h[n] = (-1/3)^n u[n-4], x[n] = (1/4)^{n+2} u[n]$$
$$h[n] = (1/4)^{n-1} u[n+3], x[n] = (1/4)^{n+2} u[n-6]$$

Make up a few on your own!