Linear Systems and Signals

Convolution with decaying exponentials

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Learning objectives

The learning objectives for this section are:

- manually compute convolutions in the time domain
- use standard formulas to simplify output calculations

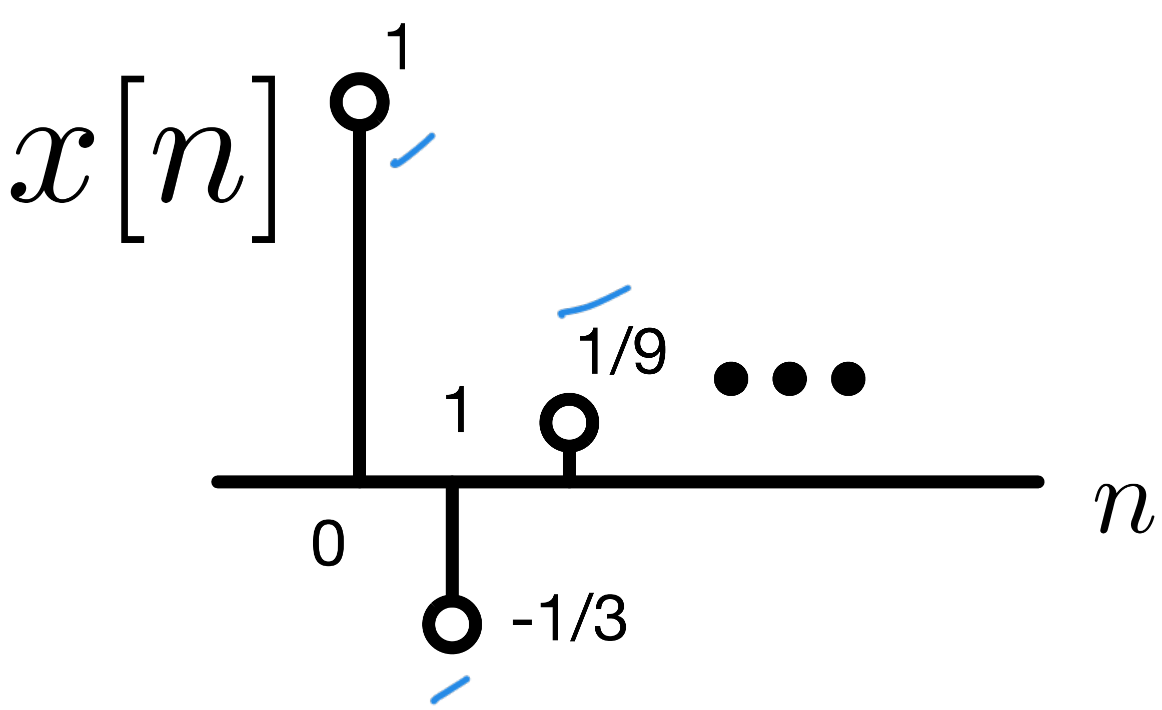
Convolving an exponential and a step

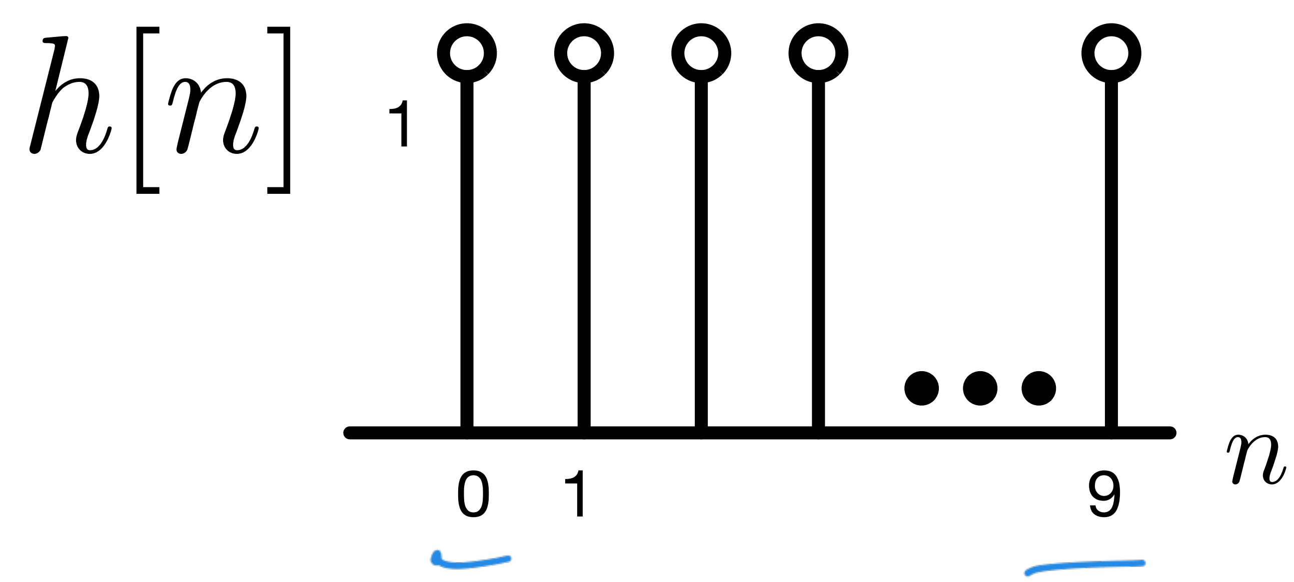
Find $y[n] = (x * h)[n]$ when

$$x[n] = \left(-\frac{1}{3}\right)^n u[n]$$

$$h[n] = u[n] - u[n-10]$$

Step 0: draw a picture and rewrite the signals if needed:





$$x[n] = \left(-\frac{1}{3}\right)^n u[n] \qquad h[n] = \sum_{m=0}^{9} \delta[n-m]$$

Writing out the convolution explicitly

1. Substitute $x[k]$ and $h[k]$ into the convolution formula. Use the step function to simplify the limits

$x[k]$ $h[n-k]$

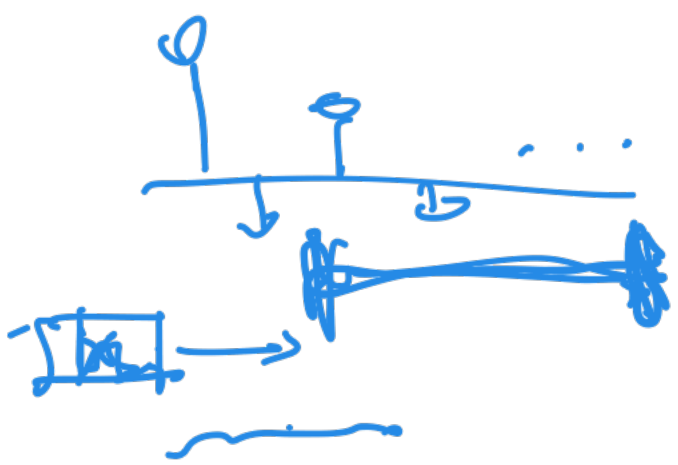
$$y[n] = \sum_{k=-\infty}^{\infty} \left(-\frac{1}{3}\right)^k u[k] \sum_{m=0}^{9} \delta[n-k-m]$$

$u[k] = 0$ if $k < 0$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{9} \delta[n-k-m] \left(-\frac{1}{3}\right)^k$$

2. For each n, $\delta[n-k-m] = 1$ only when $\boxed{k = n - m}$. Since $k \geq 0$, for $n = 0, 1, \ldots, 9$, m can run from 0 to n (so k runs from n to 0), and for $n > 9$, m can run from 0 to 9 (so k runs from n to $n-9$)

$$y[n] = \begin{cases} \sum_{k=0}^{n} \left(-\frac{1}{3}\right)^k & 0 \leq n \leq 9 \\ \sum_{k=n-9}^{n} \left(-\frac{1}{3}\right)^k & n > 10 \end{cases}$$



Some useful geometric series formulae

The geometric series summation occurs all over the place in discrete-time systems since it's the sum of a discrete-time exponential. Here are some useful formulas. We need $|\alpha| < 1$ for these series to converge.

$$\boxed{\sum_{m=0}^{\infty} \alpha^m = \frac{1}{1-\alpha}}$$

$$\sum_{m=N}^{\infty} \alpha^m = \alpha^N \sum_{m=0}^{\infty} \alpha^m = \frac{\alpha^N}{1-\alpha}$$

$$\sum_{m=0}^{M} \alpha^m = \sum_{m=0}^{\infty} \alpha^m - \sum_{m=M+1}^{\infty} \alpha^m = \frac{1-\alpha^{M+1}}{1-\alpha}$$

$$\sum_{m=M}^{N} \alpha^m = \sum_{m=0}^{N} \alpha^m - \sum_{m=0}^{M-1} \alpha^m = \frac{\alpha^M - \alpha^{N+1}}{1-\alpha}$$

Applying this to our example

4. Use the geometric series formulae:

partial overlap

$$y[n] = \sum_{k=0}^{n} \left(-\frac{1}{3}\right)^k = \frac{1 - (-1/3)^{n+1}}{1 + 1/3} \qquad 0 \le n \le 9$$

full overlap

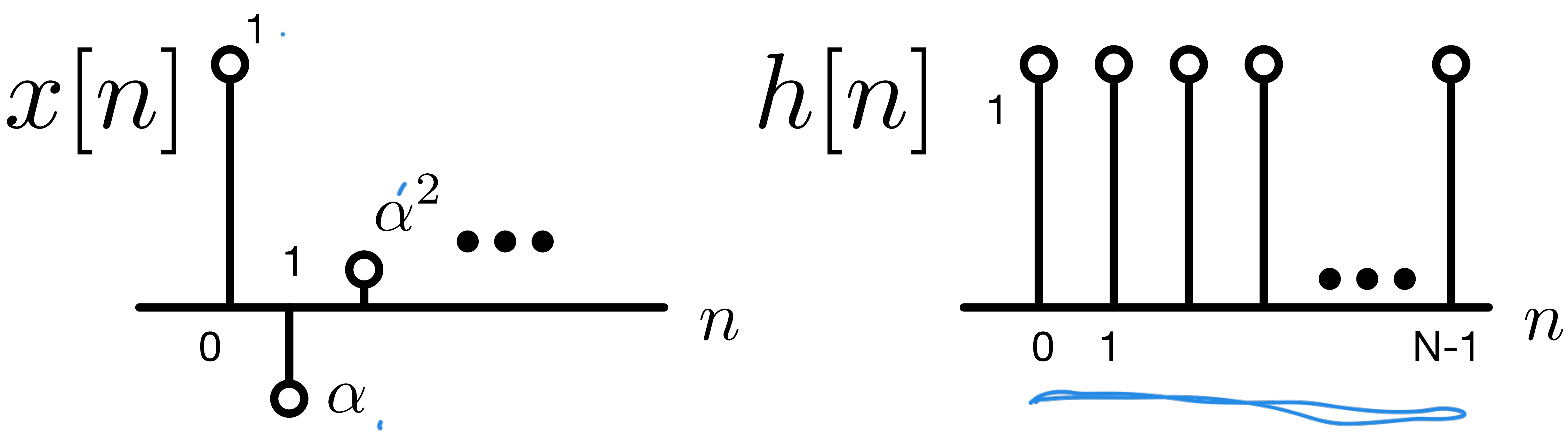
$$y[n] = \sum_{k=n-9}^{n} \left(-\frac{1}{3}\right)^k = \frac{(-1/3)^{n-9} - (-1/3)^{n+1}}{1 + 1/3} \qquad n > 9$$

Generalizing the example

Find $y[n] = (x * h)[n]$ when

$$x[n] = \alpha^n u[n]$$
$$h[n] = u[n] - u[n-N]$$

Step 0: draw a picture and rewrite the signals if needed:



$$x[n] = \alpha^n u[n] \qquad h[n] = \sum_{m=0}^{N-1} \delta[n-m]$$

Writing out the convolution explicitly

1. Substitute $x[k]$ and $h[k]$ into the convolution formula. Use the step function to simplify the limits

substitute q = N-1

$$y[n] = \sum_{k=-\infty}^{\infty} \underline{\alpha^k u[k]} \sum_{m=0}^{N-1} \underline{\delta[n-k-m]}$$

$$= \underline{\sum_{k=0}^{\infty} \sum_{m=0}^{N-1}} \delta[n-k-m]\alpha^k$$

2. For each n, $\delta[n \underline{- k -} m] = 1$ only when $\underline{k = n - m}$. Since $k \geq 0$, for $n = 0, \ldots, N-1$, m can run from 0 to n (so k runs from n to 0), and for $n \geq N$, m can run from 0 to $N-1$ (so k runs from n to $n-(N-1)$)

partial overlap

$$y[n] = \begin{cases} \sum_{k=0}^{n} \alpha^k & 0 \leq n \leq N-1 \\ \sum_{k=n-N+1}^{n} \alpha^k & n \geq N \end{cases}$$

full overlap.

Applying the general series formulas

4. Use the geometric series formulae:

$$y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha} \qquad 0 \le n \le N-1$$

$$y[n] = \sum_{k=n-N+1}^{n} \alpha^k = \frac{\alpha^{n+1-N} - \alpha^{n+1}}{1-\alpha} \qquad n \ge N$$

Try some yourself

Problem

Find the convolution from the following input-output relations:

$$h[n] = u[n] - u[n-7], x[n] = \left(-\frac{1}{3}\right)^n u[n]$$

$$h[n] = \left(\frac{1}{4}\right)^n u[n], x[n] = u[n-4]$$

$$h[n] = u[n] - u[n-3], x[n] = \left(-\frac{1}{3}\right)^n u[n-2]$$

$$h[n] = u[n-4] - u[n-8], x[n] = \left(\frac{1}{2}\right)^n u[n]$$

Make up a few on your own!