Linear Systems and Signals
DT convolution

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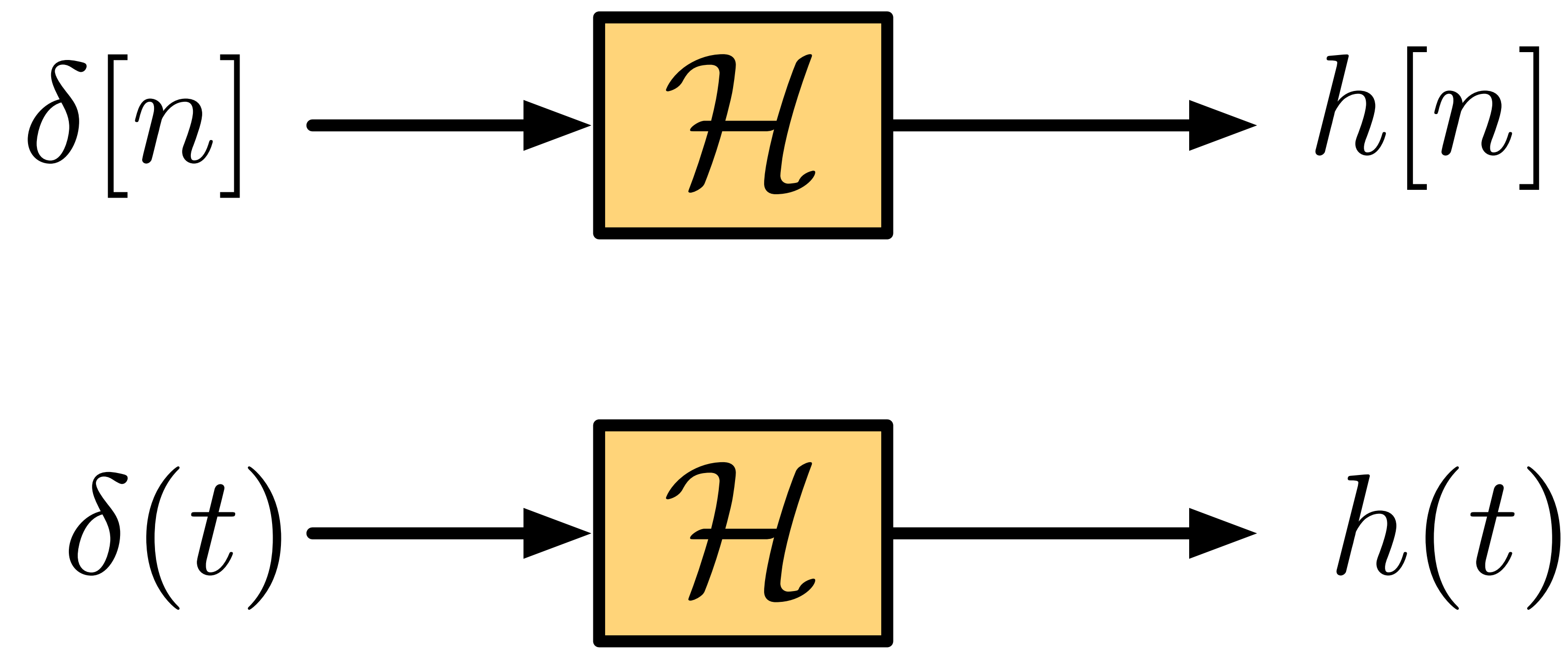
Learning objectives

The learning objective for this section is:

- show that the output of a DT LTI system is the convolution of the input with the impulse response

The impulse response



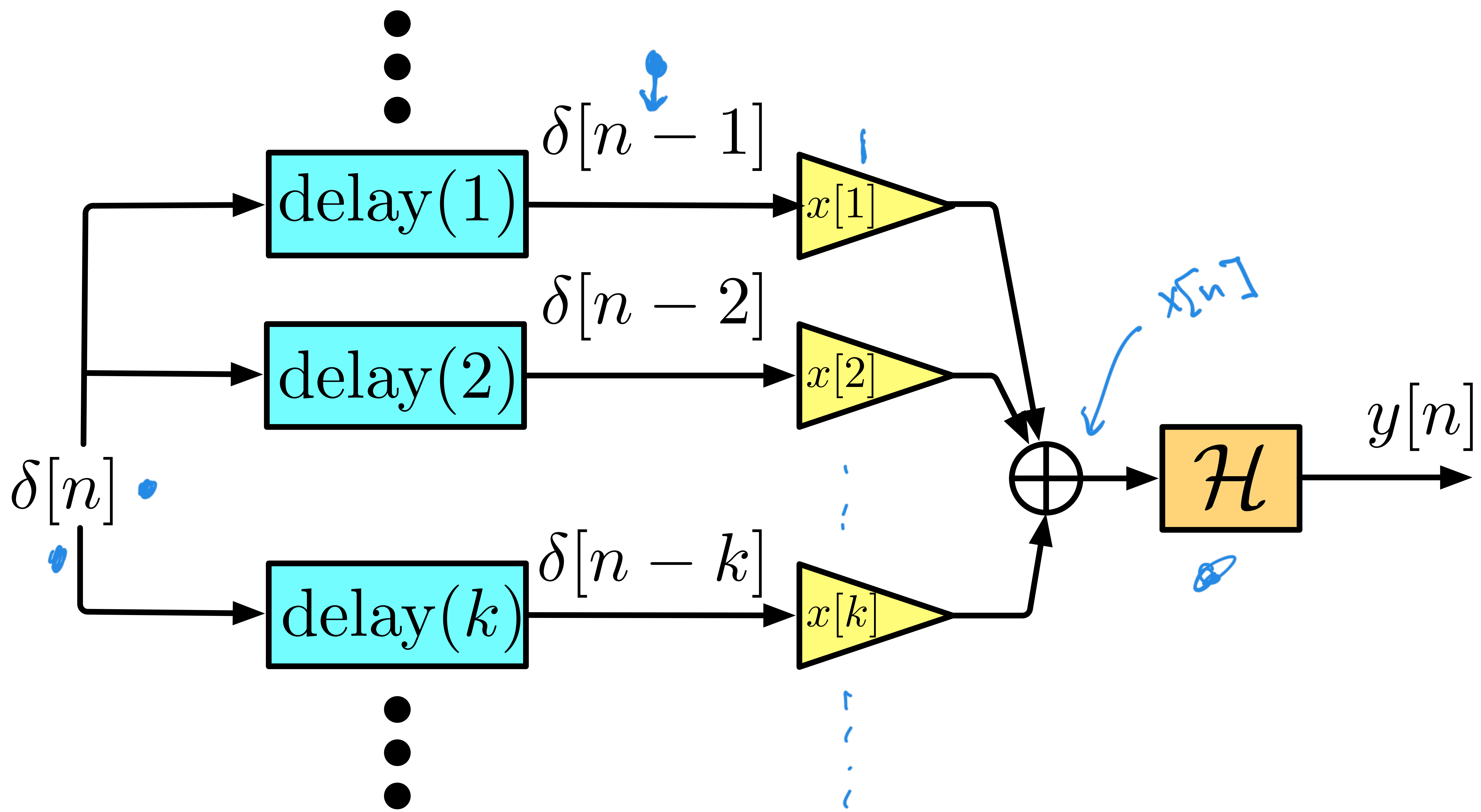
Define the *impulse response* of an LTI system to be the output signal when the input is a unit impulse ($\delta[n]$ in DT or $\delta(t)$ in CT).
We will next show that

output

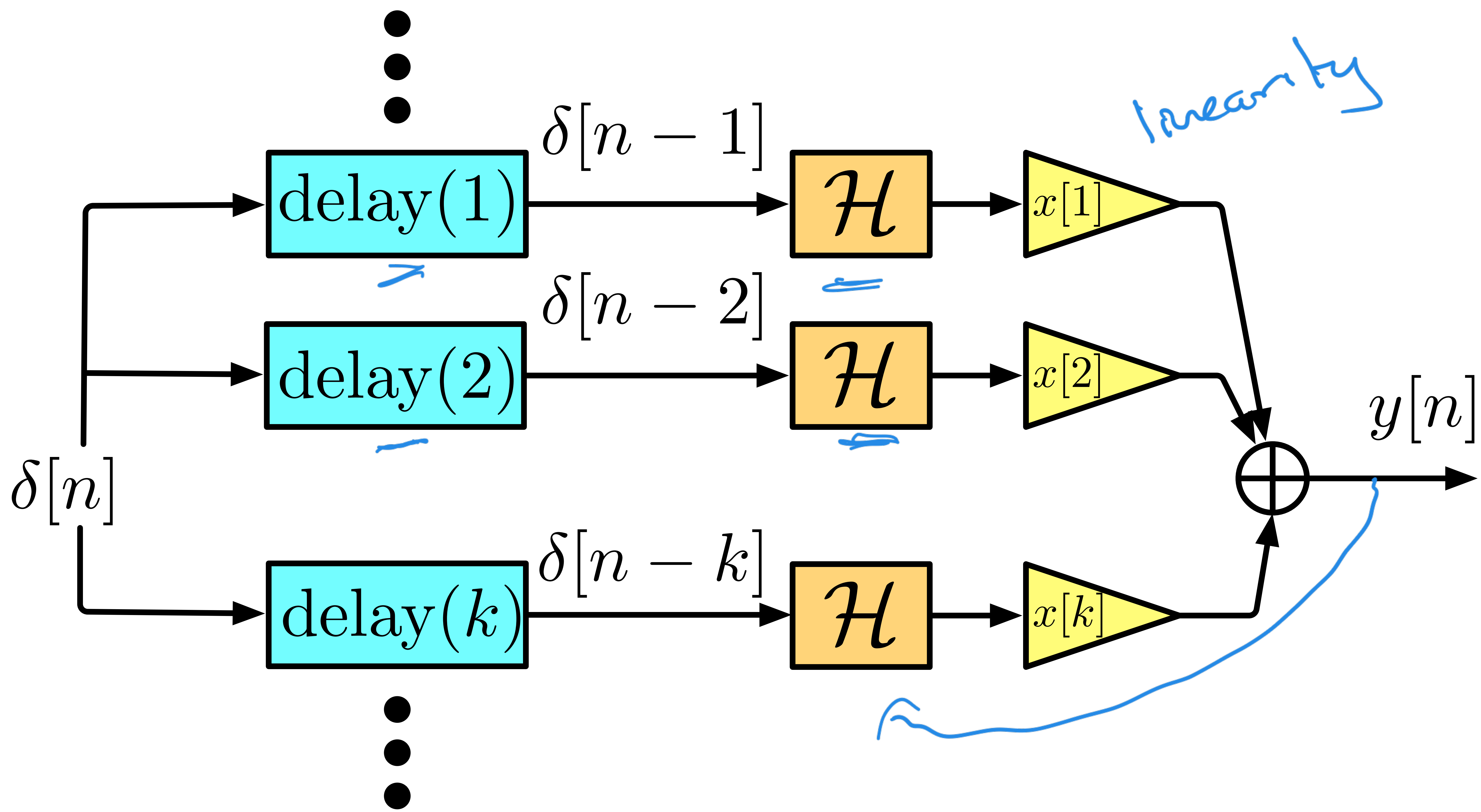
$$y[n] = \sum_{k=-\infty}^{\infty} \underline{x[k]h[n-k]}, \tag{1}$$

which is called the *DT convolution* of x and h.

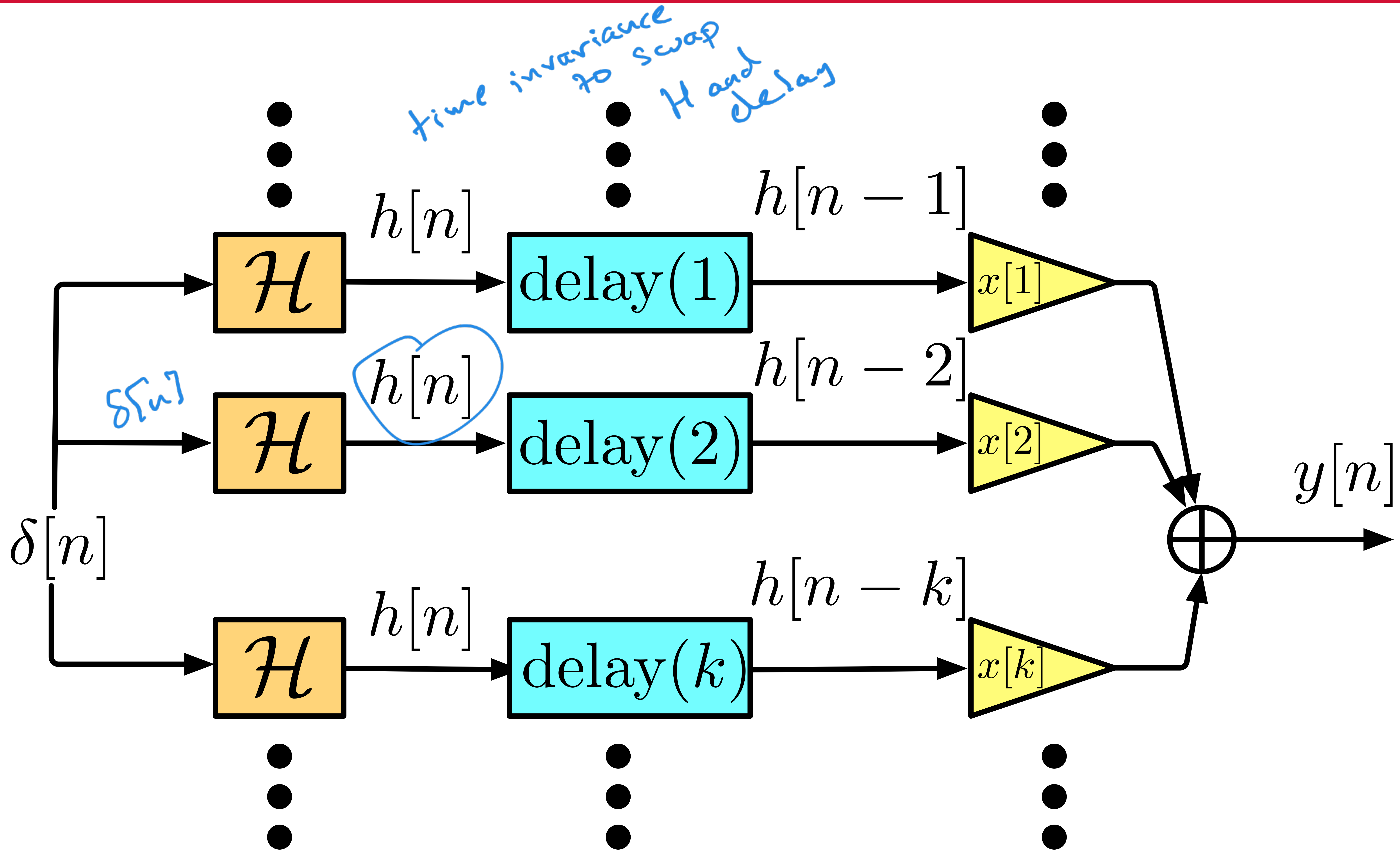
And now in pictures



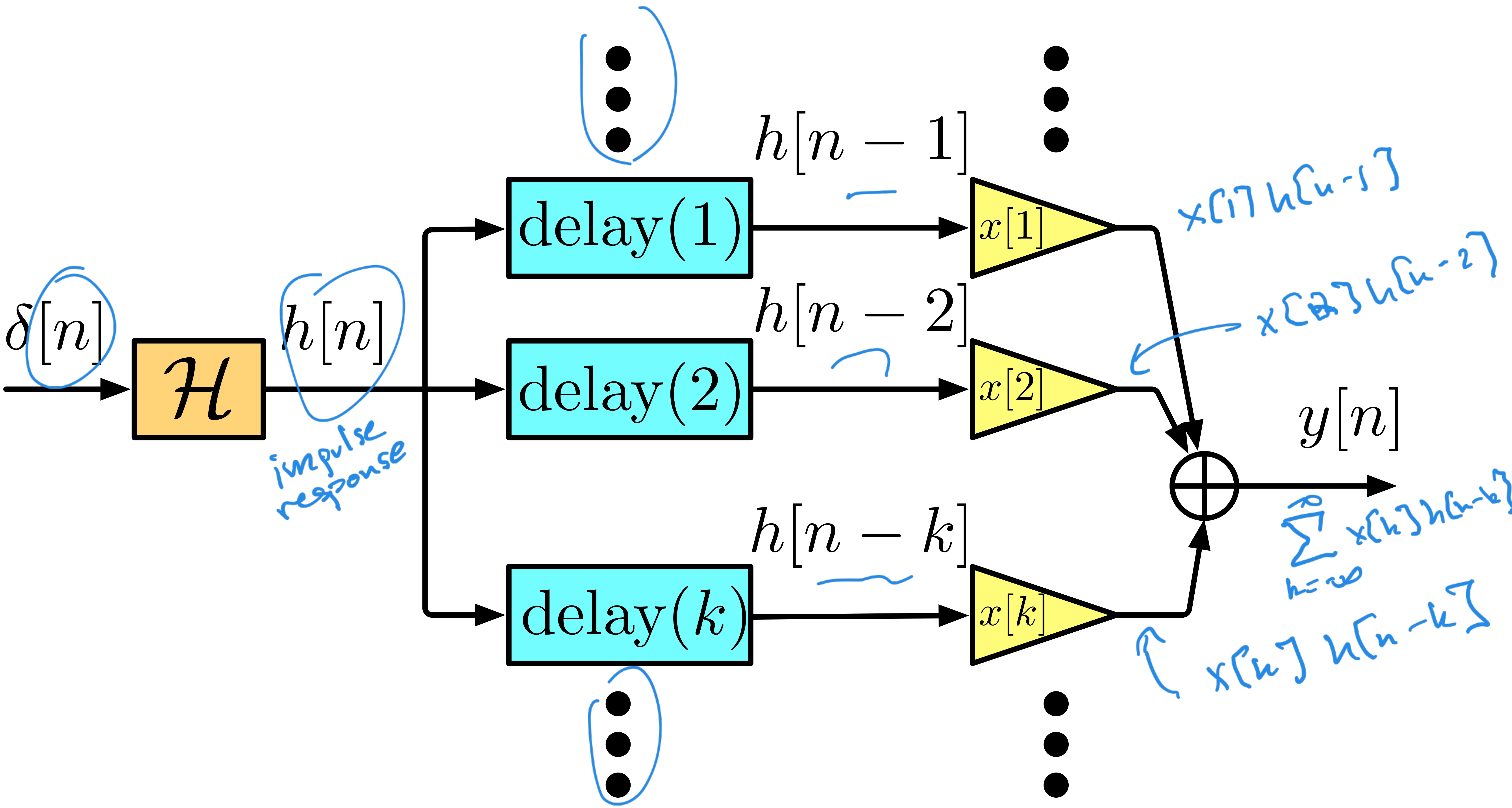
And now in pictures



And now in pictures

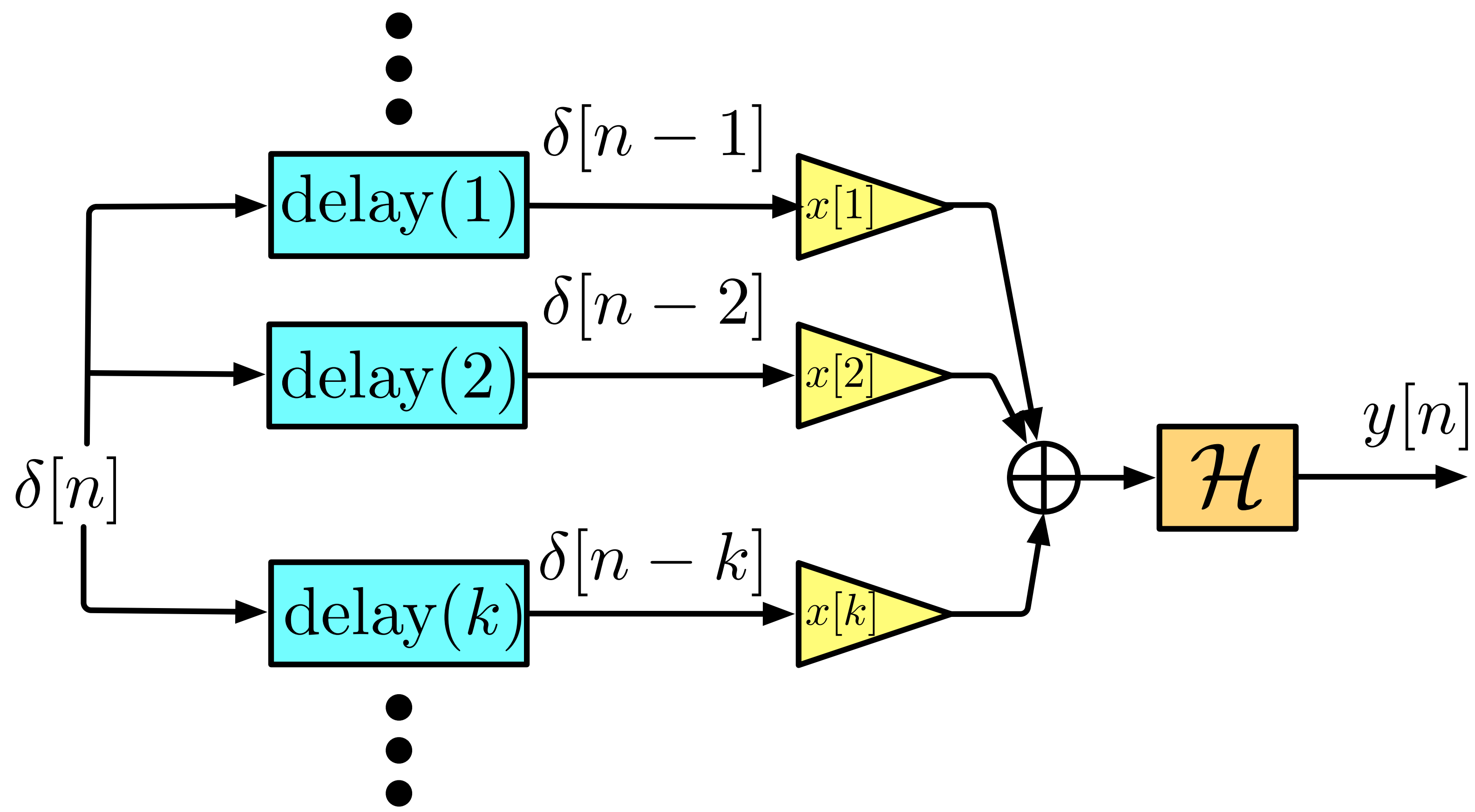


And now in pictures

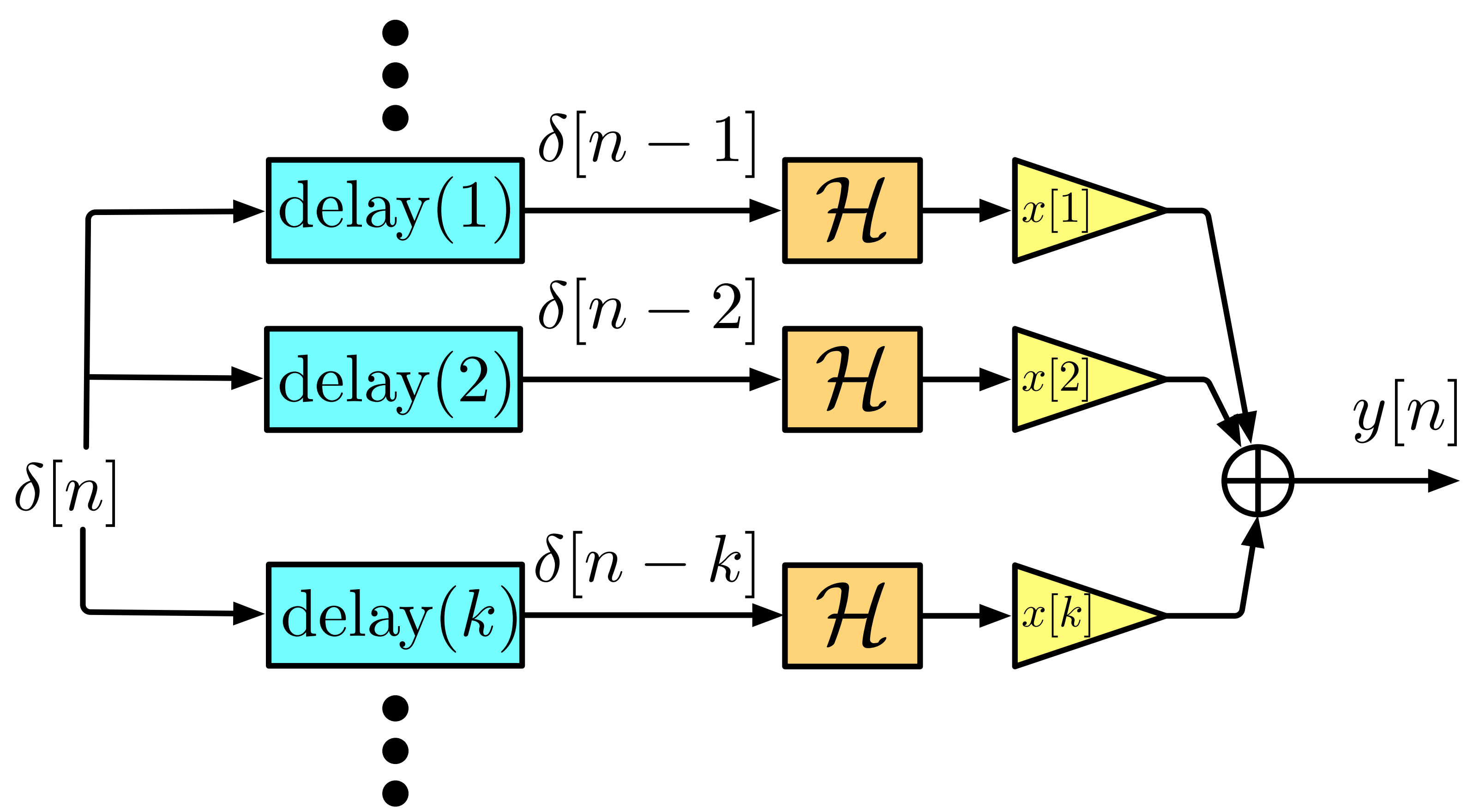


In equations

$$\mathcal{H}\left(x[n]\right) = \mathcal{H}\left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \right)$$



In equations

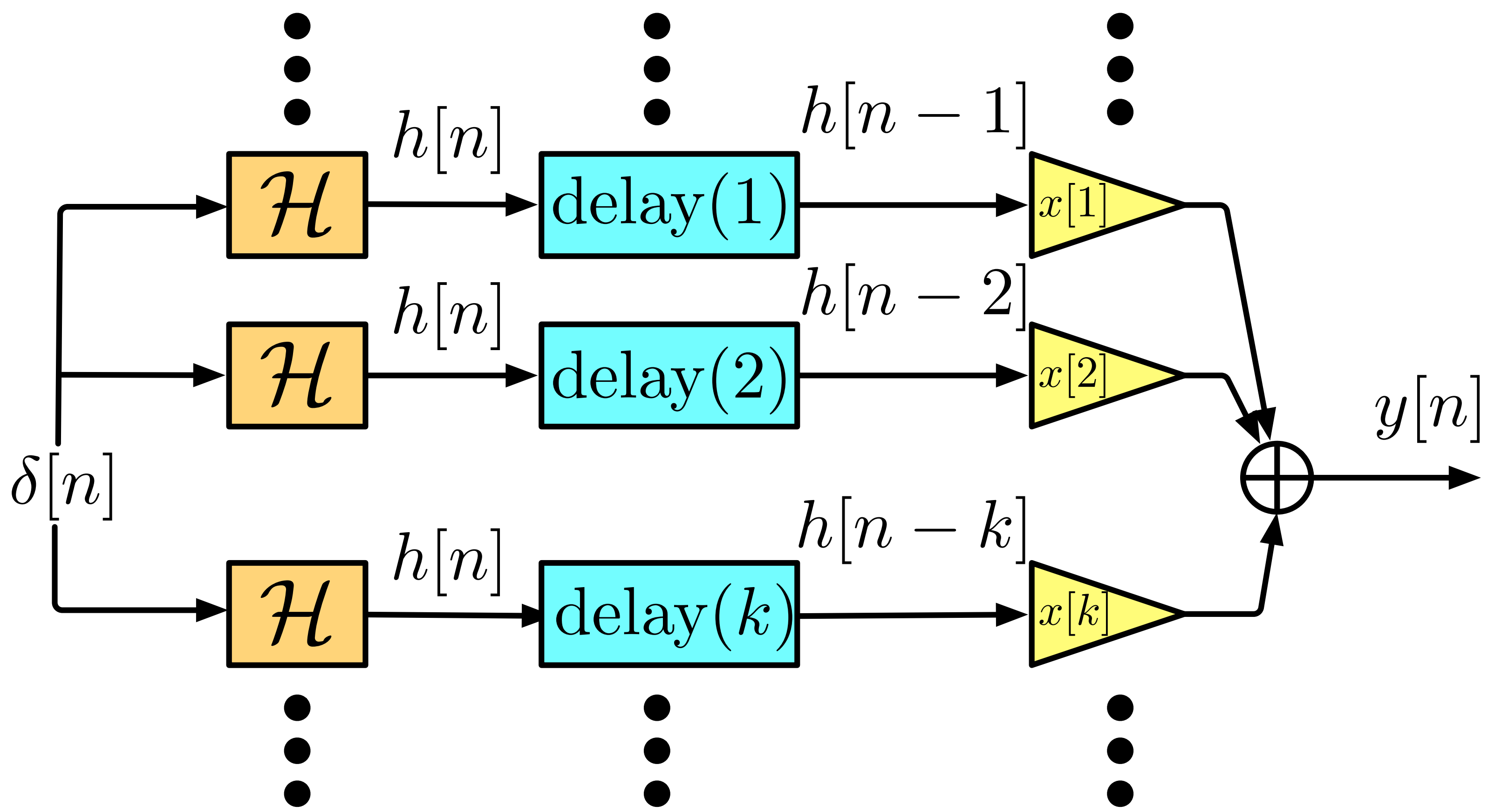


$$\mathcal{H}\left(x[n]\right) = \mathcal{H}\left(\sum_{k=-\infty}^{\infty} x[k]\underline{\delta[n-k]}\right)$$

$$\overset{\text{linearity}}{=} \sum_{k=-\infty}^{\infty} x[k]\underline{\mathcal{H}\left(\delta[n-k]\right)}$$

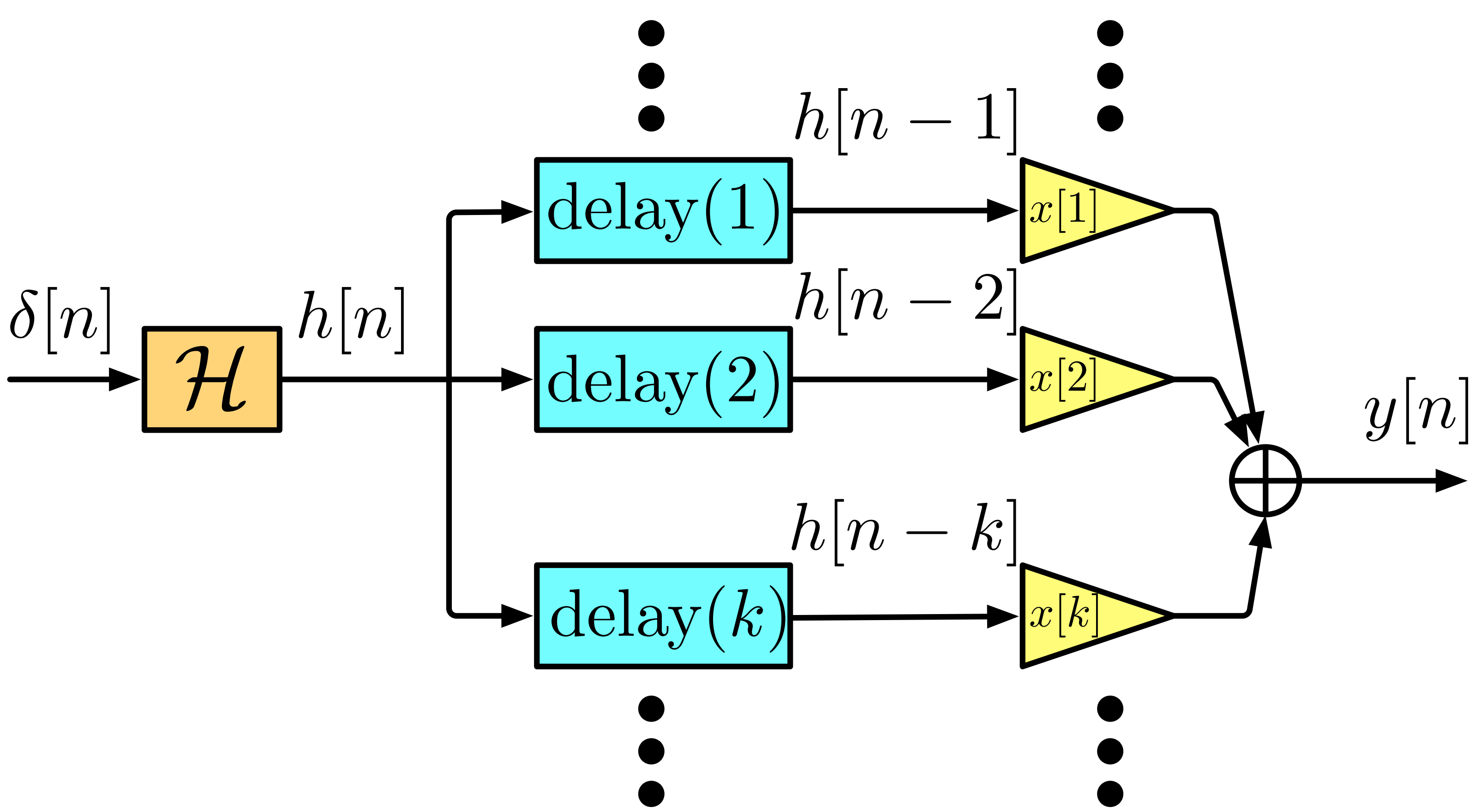
constant

In equations



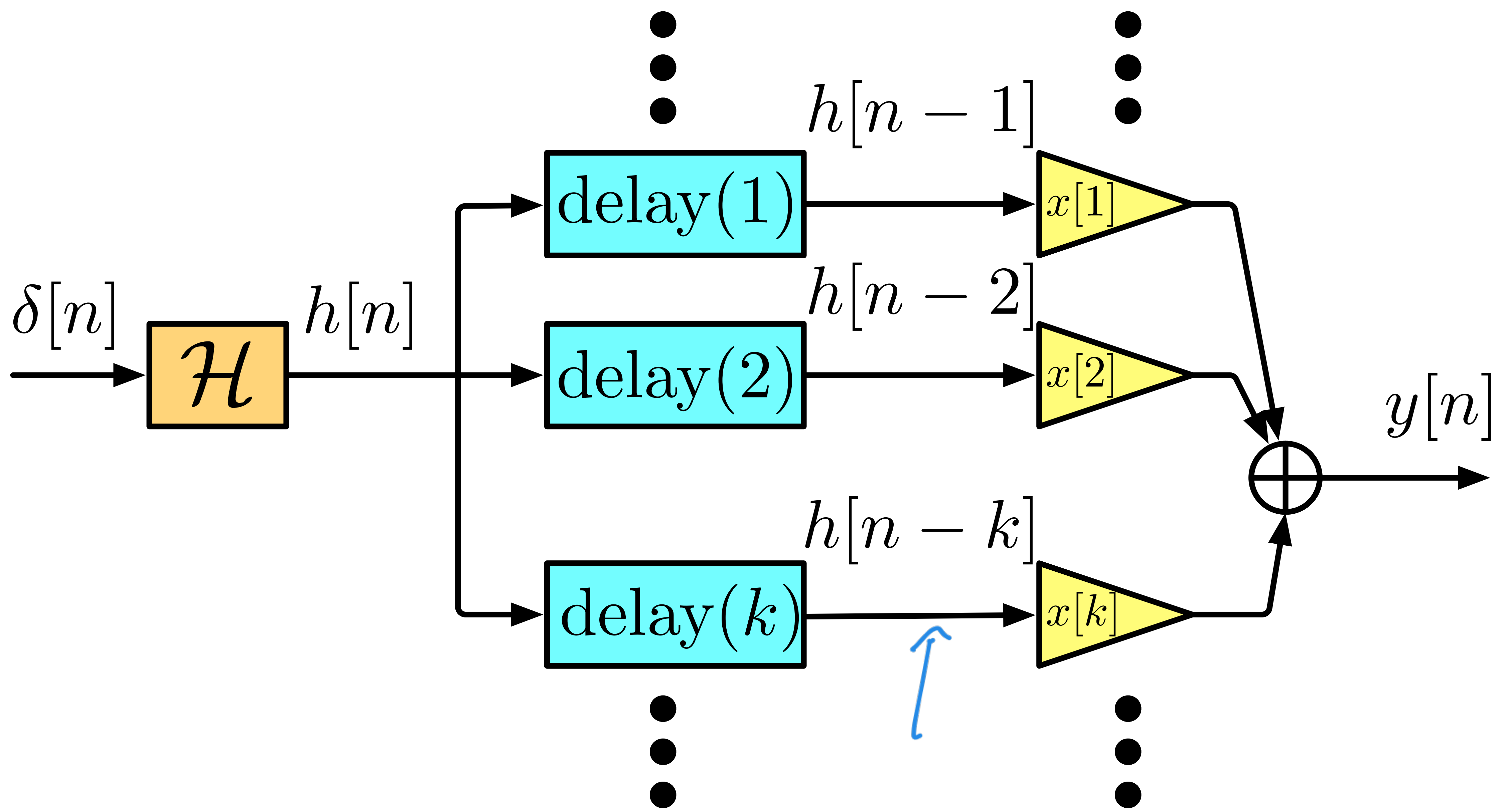
$$
\begin{aligned}
\mathcal{H}(x[n]) &= \mathcal{H}\left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \right) \\
&= \sum_{k=-\infty}^{\infty} x[k]\mathcal{H}(\delta[n-k]) \\
&= \sum_{k=-\infty}^{\infty} x[k]\mathcal{H}(\mathrm{delay}_k(\delta[n]))
\end{aligned}
$$

In equations



$$
\begin{aligned}
\mathcal{H}\left(x[n]\right) &= \mathcal{H}\left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \right) \\
&= \sum_{k=-\infty}^{\infty} x[k]\mathcal{H}\left(\delta[n-k]\right) \\
&= \sum_{k=-\infty}^{\infty} x[k]\mathcal{H}\left(\mathrm{delay}_k(\delta[n])\right) \\
&= \sum_{k=-\infty}^{\infty} x[k]\mathrm{delay}_k\left(\mathcal{H}(\delta[n])\right) \\
&= \sum_{k=-\infty}^{\infty} x[k]\underbrace{\mathrm{delay}_k\left(h[n]\right)}_{h[n-k]}
\end{aligned}
$$

Putting it together



$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \qquad (2)$$

The convolution theorem

Theorem

Let $\mathcal{H}$ be a DT linear time-invariant (LTI) system with impulse response $h[n]$. Then the output $y[n]$ to an input signal $x[n]$ is the discrete convolution of $x[n]$ and $h[n]$:

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]} \tag{3}$$

This means that the impulse response contains everything you need to know about the system.

We will often call LTI systems *filters* and talk about "the filter $h[n]$" meaning "the filter with impulse response $h[n]$."

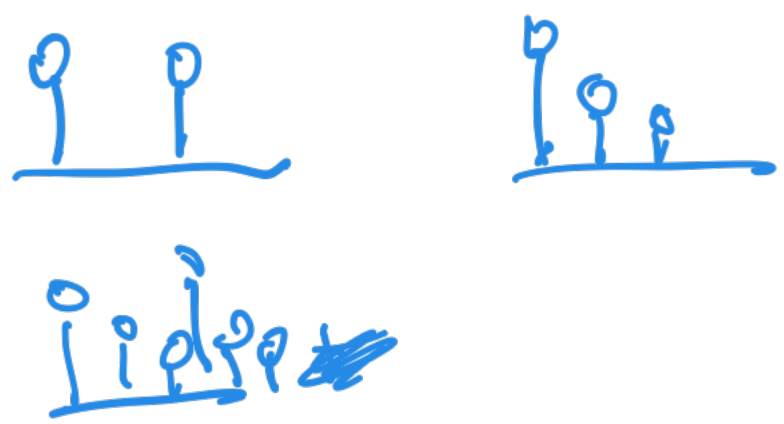
One interpretation: ringing the bell

Looking at the formula

“bell” rung delayed by k

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\underline{h[n-k]} \qquad (4)$$

two times ringing the bell



we can interpret it in the folllowing way:

- At time k, $x[k]\delta[n-k]$ enters the system.
- The system responds by copying $h[n]$ delayed by k, or $h[n-k]$ and adding it to the output.
- The output is the superposition of all these copies.

It's like at each time k the system is a bell which is hit with a force $x[k]$ and which produces $h[n]$ delayed by k.