Linear Systems and Signals

Computing the convolution using superposition

Anand D. Sarwate

Department of Electrical and Computer Engineering
Rutgers, The State University of New Jersey

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Learning objectives

The learning objective for this section is:

- calculate simple convolutions by superimposing outputs from impulses

Impulse response and convolution

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \qquad (1)$$

For "simple" signals and filters with "simple" impulse responses we can sometimes just compute the convolution directly:

1. Interpret the output as scaled and shifted copies of the impulse response $h[n]$ or as scaled and shifted copies of $x[n]$.
2. Use the "flip-and-slide" view to compute the product $h[k]x[n-k]$ for each n and add it up to get $y[n]$.
3. Write out the convolution formula and use formulas such as power series to simplify the expression for the output.

Direct calculation by superposition

Look at the formula

coefficient

h[n] delayed by k

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{2}$$

This is a sum of scaled and shifted copies of $h[n]$. Sometimes we can just write these out. For example, take $x[n] = \delta[n] - \delta[n-1]$ and $h[n] = 3\delta[n] + 2\delta[n-1] - \delta[n-2]$:

make one copy @ time n=0

subtract one copy @ n=1

$$\begin{aligned} y[n] &= 3\delta[n] + 2\delta[n-1] - \delta[n-2] && (3)\\ &\quad - 3\delta[n-1] - 2\delta[n-2] + \delta[n-3] && (4)\\ &= 3\delta[n] - \delta[n-1] - 3\delta[n-2] + \delta[n-3] && (5) \end{aligned}$$

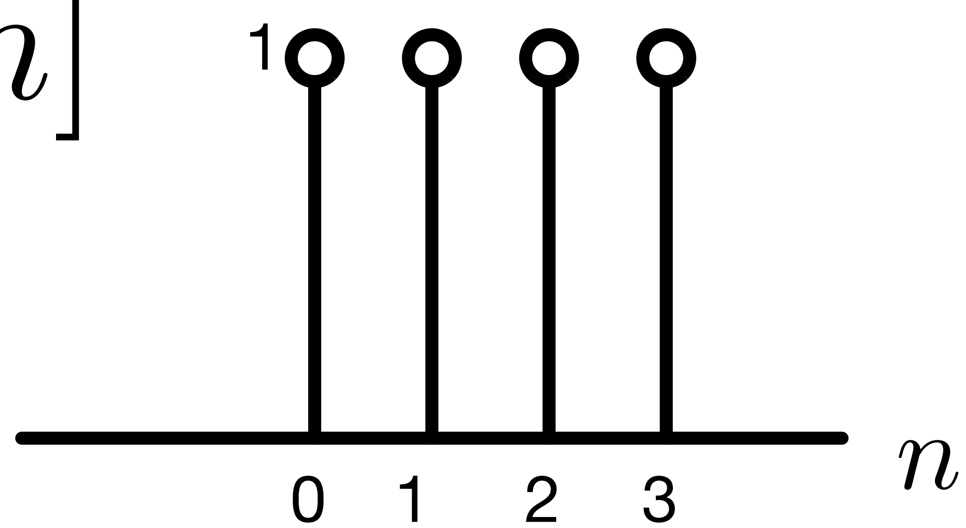
Example

Find $y[n] = (x * h)[n]$ when

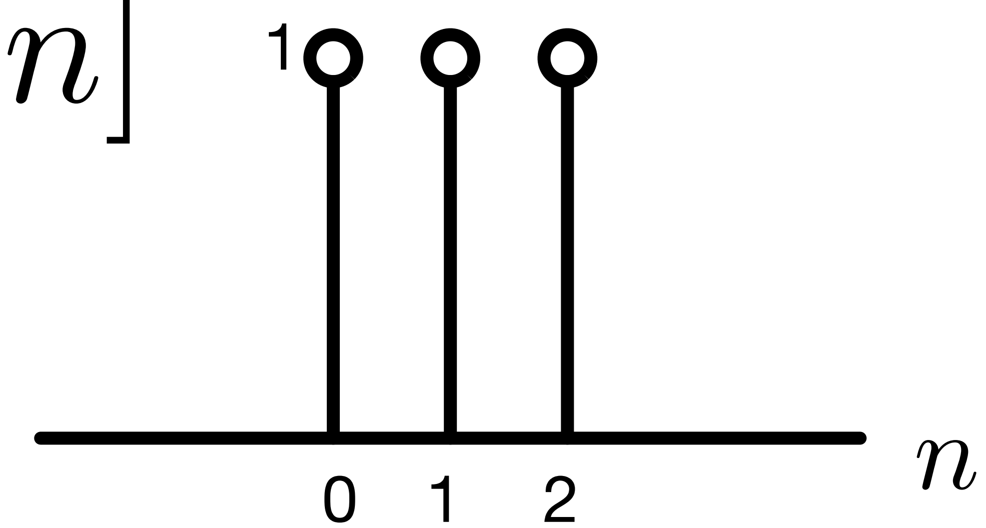
$$x[n] = u[n] - u[n-4]$$
$$h[n] = u[n] - u[n-3]$$

Step 0: draw a picture and rewrite the signals if needed:

$x[n]$



$h[n]$

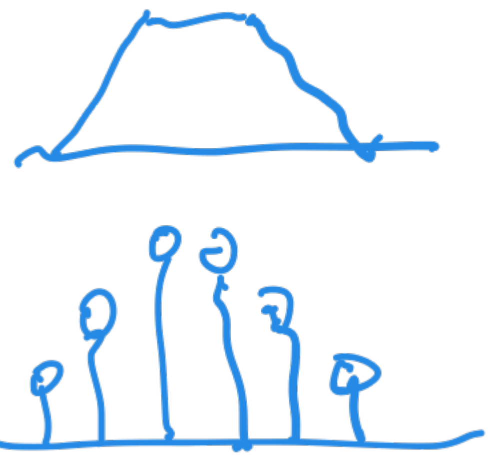


$$x[n] = \begin{cases} 1 & 0 \le n \le 3 \\ 0 & \text{otherwise} \end{cases} \qquad h[n] = \begin{cases} 1 & 0 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Direct algebraic approach

1. Write $x[n]$ and $h[n]$ in terms of δ-functions:

$$x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]$$
$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$$



2. Write scales and shifts of $h[n]$:

h[n]

h[n-1]

h[n-2]

h[n-3]

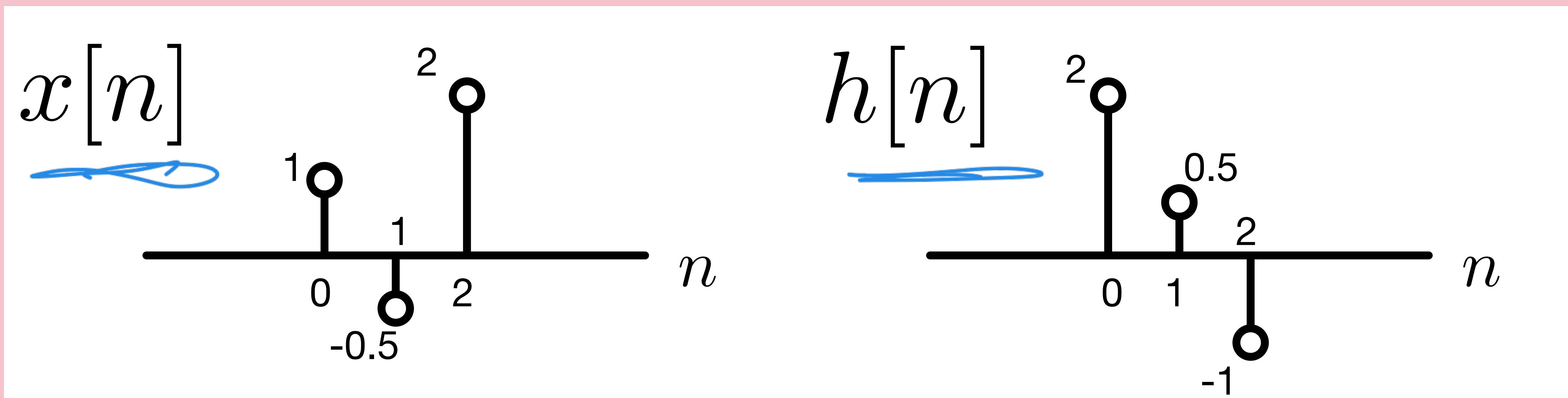
$$\begin{aligned} y[n] = \delta[n] &+ \delta[n-1] + \delta[n-2] \\ &+ \delta[n-1] + \delta[n-2] + \delta[n-3] \\ &\qquad\quad\;\; + \delta[n-2] + \delta[n-3] + \delta[n-4] \\ &\qquad\qquad\qquad\quad\;\; + \delta[n-3] + \delta[n-4] + \delta[n-5] \end{aligned}$$

3. Add them up:

$$y[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 3\delta[n-3] + 2\delta[n-4] + \delta[n-5]$$

Try some yourself

Problem



Find the convolution ~~from the following input-output relations~~:

$$h[n] = u[n] - u[n-4], x[n] = -\delta[n] + 2\delta[n-2]$$

$$h[n] = n(u[n] - u[n-3]), x[n] = 2\delta[n] - 2\delta[n-1]$$

$$h[n] = \textit{see figure above}$$

Make up a few on your own!