Linear Systems and Signals
Complex exponentials and DT LTI systems

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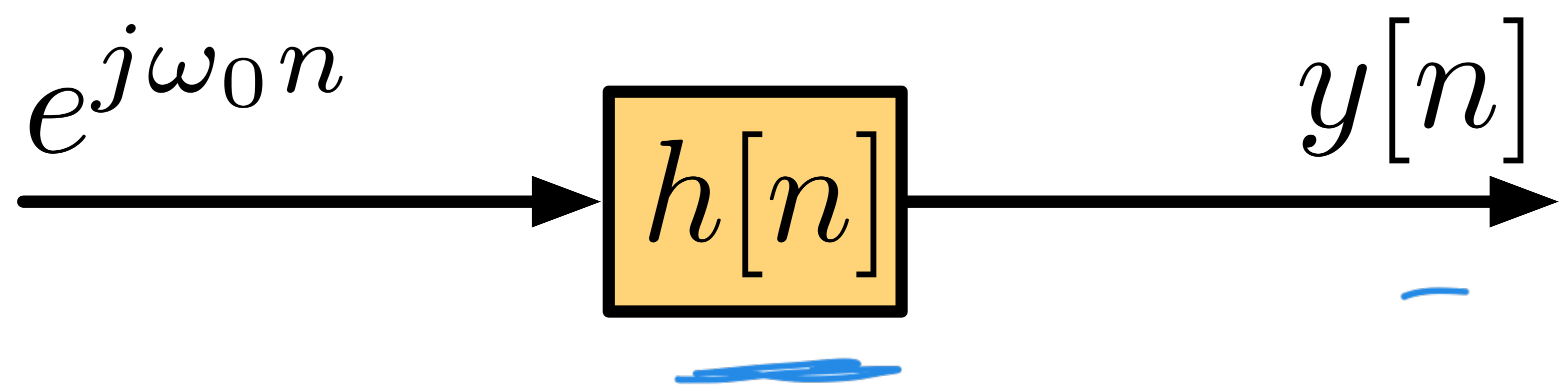
Learning objectives

The learning objectives are:

- use the eigenfunction property to compute the output of LTI systems with complex exponential input
- show that LTI systems cannot add new frequencies to the input function

Complex exponentials in DT



Recall our complex exponential signals in DT:

$$x[n] = e^{j\omega_0 n} \tag{1}$$

What happens when we pass $x[n]$ through an LTI system $h[n]$?

Convolving a complex exponential

First we have to write out the convolution:

$H(e^{j\omega_0})$

$$y[n] = \sum_{k=-\infty}^{\infty} \overbrace{e^{j\omega_0(n-k)}}^{x[n-k]} h[k] = \underbrace{e^{j\omega_0 n}}_{x[n]} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_0 k}. \qquad (2)$$

no x[n] where just a function of h[k]

Define the function

$$\boxed{H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}.} \qquad (3)$$

So $y[n]$ is the same as $x[n]$ only multiplied by a *constant* $H(e^{j\omega_0})$. Later we will see that (3) is the Discrete Fourier Transform of $h[n]$.

Eigenfunctions

$$e^{j\omega n} \longrightarrow \boxed{h[n]} \longrightarrow H(e^{j\omega})e^{j\omega n}$$

In linear algebra, there is the concept of an *eigenvector* of a square matrix. An eigenvector $\mathbf{v}$ of a matrix $\mathbf{B}$ satisfies

$$\mathbf{B}\mathbf{v} = \lambda\mathbf{v} \tag{4}$$

same v

constant

In the same way, in our vector space of functions, LTI systems act like matrices and

eigenvalue

eigenvector

$$\mathcal{H}\left(e^{j\omega n}\right) = H(e^{j\omega})e^{j\omega n} \tag{5}$$

That is, *complex exponentials are eigenfunctions of LTI systems.*

Magnitude and phase shifts

The quantity

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \tag{6}$$

is in general a complex number. So if we write $x[n]$ in magnitude-phase form,

is $\boxed{A e^{j\omega n}}$

$|A| e^{j\measuredangle A}$

$$x[n] = |A| e^{j(\omega n + \measuredangle A)} \tag{7}$$

We have

$$y[n] = |H(e^{j\omega})||A| e^{j(\omega n + \measuredangle A + \measuredangle H(e^{j\omega}))}. \tag{8}$$

That is, LTI systems cause magnitude and phase shifts to complex exponentials but *do not change their frequency*.

LTI systems cannot create new frequencies

This means that LTI systems cannot create new frequencies. If you have $x[n] = \cos(\omega_0 n)$ and a system $h[n] = \delta[n] - \delta[n-1]$ the easy solution to finding the output is to interpret $\delta[n]$ as making a copy of $x[n]$ and $\delta[n-1]$ making a copy of $\delta[n-1]$:

$$y[n] = \cos(\omega_0 n) - \cos(\omega_0 (n-1)). \tag{9}$$

Both terms have the same frequency ω_0. More generally, if

$$h[n] = \sum_{k=-\infty}^{\infty} h[k]\delta[n-k] \tag{10}$$

then

$$(h * x)[n] = \sum_{k=-\infty}^{\infty} h[k]\cos(\omega_0 (n-k)). \tag{11}$$

This looks too messy. Can we write this as a single cosine?

The eigenfunction approach

Go back to $h[n] = \delta[n] - \delta[n-1]$ and $x[n] = \cos(\omega_0 n)$. Then we can Eulerize:

$$x[n] = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n} \qquad (12)$$

Now we need

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} = 1 - e^{-j\omega}. \qquad (13)$$

Note that

phase is flipped

$$H(e^{-j\omega}) = 1 - e^{j\omega} = H(e^{j\omega})^* = |H(e^{j\omega})|e^{-j\measuredangle H(e^{j\omega})}. \qquad (14)$$

Now we can find

$$y[n] = H(e^{j\omega_0})\frac{1}{2}e^{j\omega_0 n} + H(e^{j\omega_0})^*\frac{1}{2}e^{-j\omega_0 n} \qquad (15)$$

Output has the same frequencies

We end up with

Flipped phase from H^*

$$y[n] = |H(e^{j\omega_0})|\frac{1}{2}e^{j(\omega_0 n + \measuredangle H(e^{j\omega_0}))} + |H(e^{j\omega_0})|\frac{1}{2}e^{-j(\omega_0 n + \measuredangle H(e^{j\omega_0}))}$$
$$= |H(e^{j\omega_0})| \cos(\omega_0 n + \measuredangle H(e^{j\omega_0})). \quad (16)$$

Let's take $\omega_0 = \pi/4$. Then we get $e^{-j\pi/4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j$ and

$$H(e^{j\pi/4}) = 1 - e^{-j\omega} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j \quad (17)$$

$$|H(e^{j\pi/4})| = \sqrt{2 - \sqrt{2}} \quad (18)$$

$$\measuredangle H(e^{j\pi/4}) = \tan^{-1}\frac{1}{\sqrt{2}-1} \quad (19)$$

So

$$y[n] = \left(\sqrt{2-\sqrt{2}}\right)\cos\left((\pi/4)n + \tan^{-1}\frac{1}{\sqrt{2}-1}\right). \quad (20)$$

Cosines get a magnitude and phase shift

Since the phase is an odd function, in general cosines get a magnitude and phase shift:

$$x[n] = \cos(\omega_0 n + \phi) \tag{21}$$

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\measuredangle H(e^{j\omega})} \tag{22}$$

$$y[n] = |H(e^{j\omega_0})| \cos(\omega_0 n + \phi + \measuredangle H(e^{j\omega_0})) \tag{23}$$

We call this the sinusoidal response of the system. If you can write the input signal as a linear combination of complex exponentials, the output has the same set of frequencies with different magnitudes and phases.

Recap

The main result of the preceding discussion is the following.

Theorem

Complex exponentials are eigenfunctions of LTI systems.

What this implies:

- If the input is a complex exponential, the output is also a *complex exponential at the same frequency*.
- LTI systems *cannot create* a *new* ~~*a*~~ *frequency* component that is not present in the inputs.
- Sinusoids (and complex exponentials) *get a magnitude and phase shift* when passed through an LTI system.

Try it yourself!

Problem

Find the sinusoidal responses for these combinations of input and impulse response:

$$x[n] = \cos((\pi/6)n), h[n] = \delta[n-1] \tag{24}$$

$$x[n] = \cos((\pi/6)n + \pi/4), h[n] = \delta[n] + \delta[n-2] \tag{25}$$

$$x[n] = 4\sin((\pi/8)n + \pi/3), h[n] = \delta[n-1] + \delta[n+1] \tag{26}$$

Make some up on your own!