Linear Systems and Signals
Impulse response and invertibility

Anand D. Sarwate

Department of Electrical and Computer Engineering
Rutgers, The State University of New Jersey

2020

Learning objectives

The learning objective for this section is:

- determine if a system is invertible using the impulse response

Invertibility and the impulse response

A system $\mathcal{G}$ is called an inverse system for a system $\mathcal{H}$ if $\mathcal{G}(\mathcal{H}(x)) = x$. What happens when the systems are LTI? For all $x(t)$ or $x[n]$,

$$(g * (h * x))(t) = x(t) \qquad ((g * (h * x))[n] = x[n] \tag{1}$$

Note that $x(t) * \delta(t) = x(t)$ and $x[n] * \delta[n] = x[n]$. Since convolution is *associative*, we want to find g such that

$$(g * h)(t) = \delta(t) \qquad (g * h)[n] = \delta[n] \tag{2}$$

When is it possible to have an LTI system g invert a system h?

An example

Suppose $h[n] = \left(\frac{1}{3}\right)^n u[n]$. This is the impulse response of a system with a recursive input-output relation:

$$y[n] = x[n] + \frac{1}{3}y[n-1]. \quad = x[n] + \frac{1}{3}x[n-1] + \frac{1}{9}y[n-2] \tag{3}$$

Solving for $x[n]$, to get a system with input $y[n]$ and output $x[n]$,

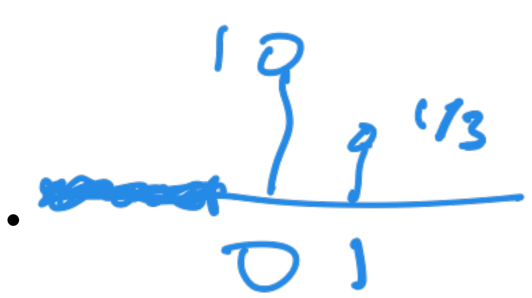
$x[n] \to \boxed{h[n]} \to y[n]$

$$x[n] = y[n] - \frac{1}{3}y[n-1]. \tag{4}$$

$y[n] \to \boxed{g[n]} \to x[n]$

So this has impulse response

$$g[n] = \delta[n] - \frac{1}{3}\delta[n-1]. \tag{5}$$



$1 + 1/3 < \infty$

This inverse is stable and causal.

What if the inverse is unstable?

What happens if g is unstable? Then we might be in a situation where $(h * x)(t)$ is one of the input signals that causes the output of g to blow up. For example, take the differentiator:

$$y(t) = \frac{d}{dt} x(t). \tag{6}$$

The inverse system is an integrator, which we know is unstable. Should we build the inverse system?

Similarly if $h(t) = u(t)$ (which is an integrator), the output $y(t)$ could become unbounded so we would need $g(t)$ to be able to handle an arbitrarily large input. Is that ok?

What if the inverse is noncausal?

Let's look at a situation where $g[n]$ might have be noncausal: take $h[n]$ to be a simple delay $\delta[n - n_0]$. The inverse system has impulse response $\delta[n + n_0]$ representing a rewind of n_0 steps. But this is a noncausal system. We can also find more complicated examples where the inverse system has a nonzero impulse response for all $n < 0$.

Are noncausal systems ok? This means that we have to wait for all of the output $y[n]$ to be generated (possibly infinite time) before applying $g[n]$ to get back $x[n]$. Is that ok?

Realizability

Often we want the inverse system to be both causal and stable. Sometimes this is called a *realizable* system – it can, in real time, invert the given system $h[n]$.

The important thing to remember is that while we might be able to write down an inverse system, in order to really invert the system in practice we need the inverse to be stable and causal.

We will discuss and do example problems for invertibility of LTI systems more when we talk about transforms.