Linear Systems and Signals

Impulse response and stability

Anand D. Sarwate

Department of Electrical and Computer Engineering
Rutgers, The State University of New Jersey

2020

Learning objectives

The learning objective for this section is:

- determine if a system is stable using the impulse response

Stability and the impulse response

We saw how causality relates to the impulse response, but what about stability? A system is *bounded-input bounded-output (BIBO) stable* if for any input x satisfying $|x(t)| \leq B$ for all t (for CT) or $|x[n]| \leq B$ for all n (for DT) there is a B' such that $|y(t)| \leq B'$ for all t (for CT) or $|y[n]| \leq B'$ for all n (for DT).

So the question is: how does this relate to the impulse response? Again, we look at the convolution:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \qquad y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \tag{1}$$

A sufficient condition for stability

Suppose $|x(t)| \leq B$ for all t. Let's look at bounding the magnitude of $y(t)$:

convolution

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \right| \tag{2}$$

$$\leq \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)|d\tau \tag{3}$$

$\leq B$

$$\leq B \int_{-\infty}^{\infty} |h(\tau)|d\tau \tag{4}$$

So a *sufficient condition* for $|y(t)|$ to be bounded is if the impulse response is absolutely integrable:

$$\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty \tag{5}$$

Necessary and sufficient

If a system is stable does it mean that

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty? \tag{6}$$

Let's look at a particular input $x(t) = \frac{h^*(-t)}{|h(-t)|}$ which is bounded if $h(t)$ is bounded. The output at time $t = 0$ is

$x(-\tau) = \frac{h^*(\tau)}{|h(\tau)|}$

$t=0$

$$y(0) = \int_{-\infty}^{\infty} h(\tau) x(0 - \tau) d\tau \tag{7}$$

$$= \int_{-\infty}^{\infty} h(\tau) \frac{h^*(\tau)}{|h(\tau)|} d\tau = \int_{-\infty}^{\infty} |h(\tau)| d\tau. \tag{8}$$

So for h to be stable we need $\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$: stability implies (6).

The result

An LTI system is stable *if and only if*

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty \qquad \sum_{n=-\infty}^{\infty} |h[n]| < \infty \tag{9}$$

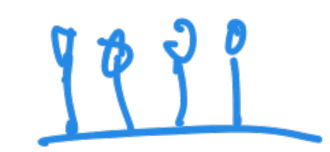
Using the same arguments, prove the condition for the DT case.

Unlike general systems, to prove stability we just have to compute this integral/sum.

Examples for stability

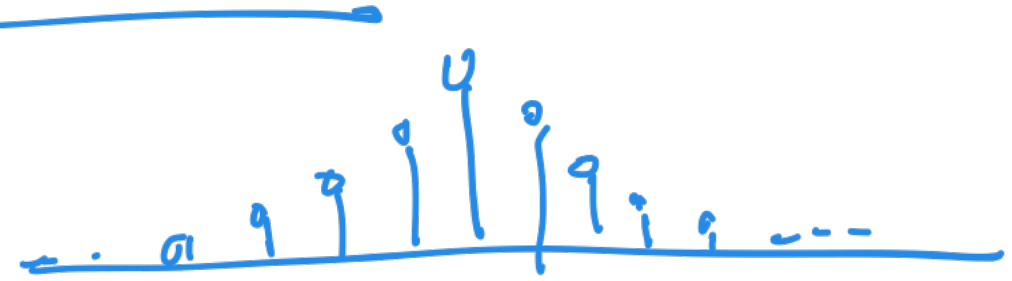
$|u(t)| = u(t)$

integrator system is unstable.

- Suppose $\cancel{x}\,h(t) = u(t)$. Since $\int_0^\infty 1 dt = \infty$, the system is unstable.
- Suppose $\cancel{x}\,h(t) = e^{-2t} u(t)$. We see that $\int_0^\infty e^{-2t} dt = \frac{1}{2}$ so the system is stable.
- Suppose $h[n] = u[n] - u[n-5]$. This is a finite-length bounded signal so $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ and the system is stable. 

$T=4$

- Suppose $h[n] = \left(\frac{1}{3}\right)^{|n|}$. This is positive and summable so it is stable.



Try it yourself

Problem

Determine whether the following LTI sytems are stable or unstable.

$$h(t) = e^{-2t}u(t+50) \tag{10}$$

$$h(t) = te^{-t}u(t) \tag{11}$$

$$h[n] = n^2 u[n-2] \tag{12}$$

$$h[n] = (0.2)^n u[n-7] \tag{13}$$

(Some problems borrowed from Oppenheim and Willsky).