

# Linear Systems and Signals

## Eigenfunctions of CT LTI systems

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# Learning objectives

The learning objectives for this section are:

- use the eigenfunction property to compute the output of LTI systems with complex exponential input
- show that LTI systems cannot add new frequencies to the input function



# Eigenvectors and eigenfunctions

Remember that for DT systems we saw that *complex exponentials are eigenfunctions of LTI systems*. The same property holds for CT LTI systems. Suppose  $x(t) = e^{j\omega_0 t}$ . Then

*convolution*

$$\int_{-\infty}^{\infty} h(\tau) e^{j\omega_0(t-\tau)} d\tau = e^{j\omega_0 t} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega_0 \tau} d\tau \quad (1)$$

*mag 1 phase =  $\omega_0 t$*

*just a function of  $h$  and  $\omega_0$*

If we define

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau \quad \text{CTFT} \quad (2)$$

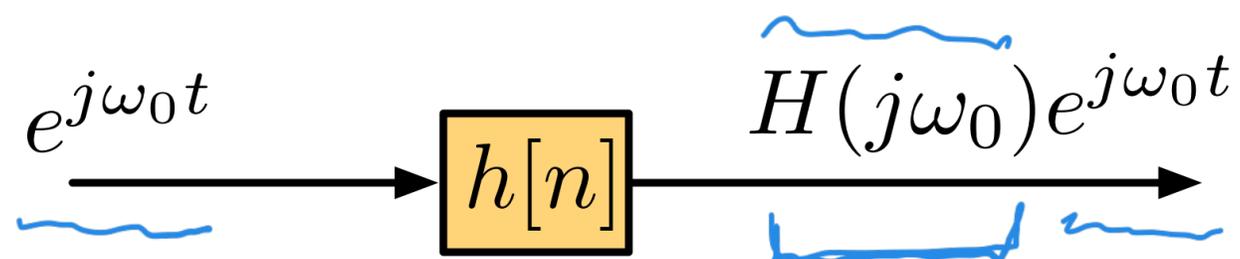
we get

$$\underline{h(t) * e^{j\omega_0 t}} = \underline{H(j\omega_0) e^{j\omega_0 t}} \quad (3)$$

$H(j\omega)$  is the *Continuous Time Fourier Transform* (more on that later).



# How a system acts on a complex exponential



The quantity

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \quad (4)$$

is a complex-valued function, so putting it in magnitude-phase form

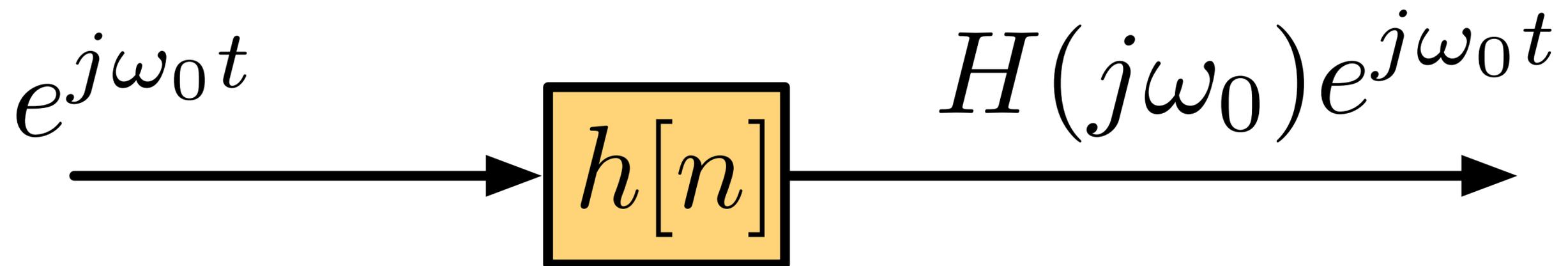
$|H(j\omega)| e^{j\angle H(j\omega)}$ , for  $x(t) = e^{j\omega_0 t}$  we get

$$(h * x)(t) = |H(j\omega_0)| e^{j(\omega_0 t + \angle H(j\omega_0))} \quad (5)$$

That means the LTI system changes the magnitude and phase of a complex exponential but not the frequency.



# Implications



- LTI systems multiply complex exponential signals by a (complex-valued) constant. This is why we say complex exponentials are eigenfunctions of LTI systems.
- LTI systems cannot create new frequencies not present in the original signal.
- Linear combinations of complex exponentials (example: cosines) produce linear combinations of complex exponentials at the same frequencies.



# Symmetry

Suppose  $h(t)$  is a real-valued signal. Take the complex conjugate of  $H$ :

$$\underline{H^*(j\omega)} = \left( \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \right)^* \quad (6)$$

$$= \int_{-\infty}^{\infty} \underline{h^*(t)} e^{j\omega t} dt \quad h^*(t) = h(t) \quad (7)$$

$$= \underline{H(-j\omega)} \quad (8)$$

So as a function of  $\omega$ ,  $H(j\omega)$  is conjugate symmetric. This implies that

$$\underline{|H(j\omega)|} \text{ is even} \quad \underline{\angle H(j\omega)} \text{ is odd} \quad (9)$$

Prove that the DT version  $H(e^{j\omega})$  is also conjugate symmetric with even magnitude and odd phase.



# An example

Suppose  $h(t) = e^{-3t}u(t)$ . Then

$$H(j\omega) = \int_{-\infty}^{\infty} \underbrace{e^{-3t}} \underbrace{e^{-j\omega t}} \underbrace{u(t)} dt \quad (10)$$

$$= \int_0^{\infty} e^{-(3+j\omega)t} dt \quad (11)$$

$$= \frac{1}{3+j\omega} \quad (12)$$

$$= \frac{3}{9+\omega^2} - j \frac{\omega}{9+\omega^2} \quad (13)$$

$$= \frac{1}{9+\omega^2} e^{-j \tan^{-1}(\omega/3)} \quad (14)$$

$$\frac{3^2 + \omega^2}{(9 + \omega^2)^2} = \frac{1}{9 + \omega^2}$$

This means an input  $e^{j\omega_0 t}$  gets a gain of  $\frac{1}{9+\omega_0^2}$  and a phase shift of  $-\tan^{-1}(\omega_0/3)$ .



# Example continued

Suppose  $h(t) = e^{-3t}u(t)$  and  $x(t) = \cos(6\pi t)$ . Eulerizing  $x(t)$ :

$$x(t) = \frac{1}{2}e^{j6\pi t} + \frac{1}{2}e^{-j6\pi t} \quad (15)$$

So now the output  $y = (x * h)(t)$ , we have (since  $\omega_0 = +6\pi$  for the first term and  $-6\pi$  for the second term) :

$$y(t) = \frac{1}{2} \frac{1}{9 + 36\pi^2} e^{j6\pi t - j \tan^{-1} \frac{1}{2\pi}} + \frac{1}{2} \frac{1}{9 + 36\pi^2} e^{-j6\pi t - j \tan^{-1} \frac{1}{2\pi}} \quad (16)$$

$$y(t) = \frac{1}{9 + 36\pi^2} \left( \frac{1}{2} e^{j(6\pi t - \tan^{-1} \frac{1}{2\pi})} + \frac{1}{2} e^{-j(6\pi t - \tan^{-1} \frac{1}{2\pi})} \right) \quad (17)$$

$$= \frac{1}{9 + 36\pi^2} \cos \left( 6\pi t - \tan^{-1} \frac{1}{2\pi} \right) \quad (18)$$



# The general case

For general  $h(t)$  and  $x(t) = \cos(\omega t)$  we get

$$y(t) = \underline{H(j\omega)} \left( \underline{\frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t}} \right) \text{ Eulerize} \quad (19)$$

$$\text{mag even} \quad \text{phase odd} = |H(j\omega)| \frac{1}{2} e^{j(\omega t + \angle H(j\omega))} + |H(-j\omega)| \frac{1}{2} e^{-j(\omega t + \angle H(-j\omega))} \quad (20)$$

$$= |H(j\omega)| \frac{1}{2} e^{j(\omega t + \angle H(j\omega))} + |H(j\omega)| \frac{1}{2} e^{-j(\omega t - \angle H(j\omega))} \quad (21)$$

$$= |H(j\omega)| \cos(\omega t + \angle H(j\omega)). \quad (22)$$

This is called the sinusoidal response of the system.

Check for yourself: what is the formula for an input of  $\sin(\omega t)$ ?



# Try it yourself

## Problem

*Find the sinusoidal responses for these combinations of input and impulse response:*

$$x(t) = \cos(100\pi n), h(t) = e^{-4t+2}u(t) \quad (23)$$

$$x(t) = \cos(4\pi t), h(t) = \delta(t - 3) \quad (24)$$

$$x(t) = 4 \sin(8\pi t + \pi/3), h(t) = e^{-2t}u(t) \quad (25)$$

*Make up some on your own!*

