Linear Systems and Signals
Algebraic calculation of convolutions

Anand D. Sarwate

Department of Electrical and Computer Engineering
Rutgers, The State University of New Jersey

2020

Learning objectives

The learning objectives for this section are:

- manually compute CT convolutions in the time domain
- simplify calculations using LTI properties and standard formulas

Reusing convolution formulas

If we know

$$y(t) = x(t) * h(t), \tag{1}$$

we can use properties of convolution/LTI systems to calculate:

delay by τ delay by τ

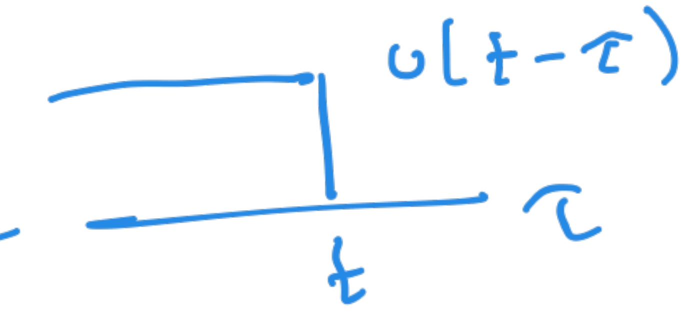
$$x(t-\tau) * h(t) = y(t-\tau) \tag{2}$$

$$x(t) * \underline{\alpha} h(t) = \underline{\alpha} y(t). \tag{3}$$

So once we've computed a "template" $y(t)$ we can use these properties to compute convolutions involving delays and gains of our $x(t)$ and $h(t)$ signals.

Example: exponential and a step

We can often calculate convolutions just using calculus:



$$\underbrace{e^{-at}u(t)}_{x(t)} * \underbrace{u(t)}_{h(t)} = \int_{\tau=-\infty}^{\infty} e^{-a\tau} u(\tau) u(t-\tau) d\tau \tag{4}$$

apply step functions

show this system is an integrator

$$y(t) = \int_{\tau=0}^{t} e^{-a\tau} d\tau \tag{5}$$

$$= \frac{1}{a}(1 - e^{-at})u(t). \tag{6}$$

signal starts at $t=0$

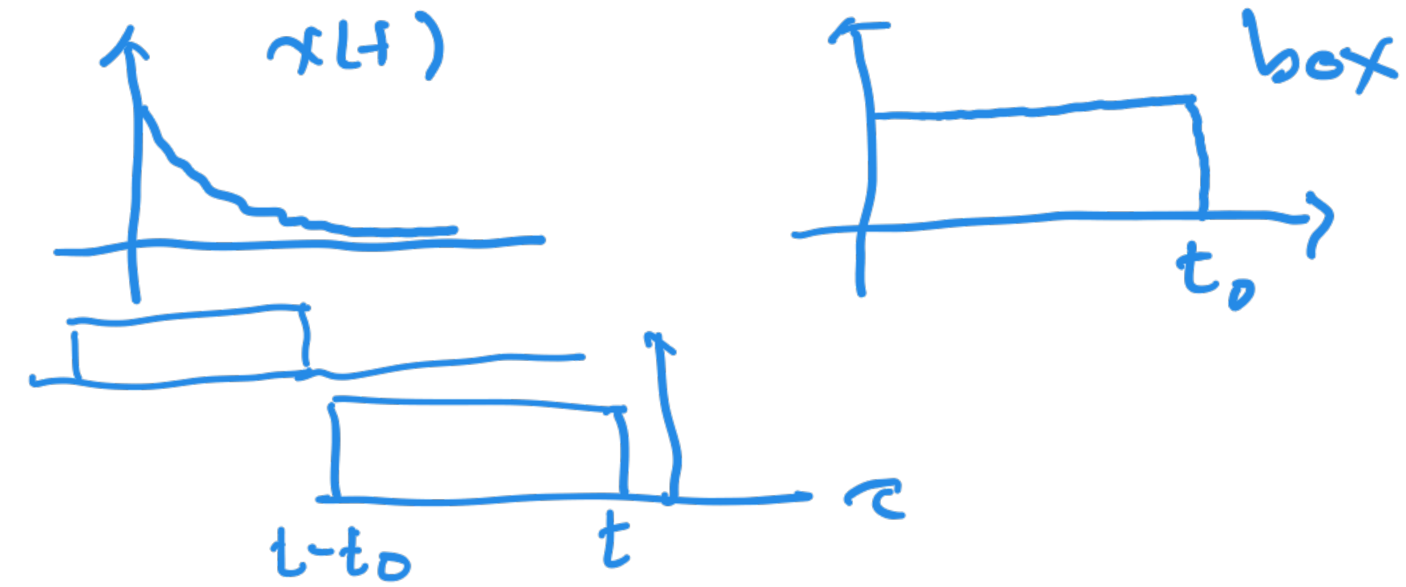
Now that we have this, we can calculate others using properties of LTI systems:

gain by 3   delay by 3

$$3e^{-at}u(t) * u(t-3) = \frac{3}{a}(1 - e^{-a(t-3)})u(t-3) \tag{7}$$

Example: exponential and difference of steps

A slightly more complicated example



$$e^{-at}u(t) * (u(t) - u(t - t_0))$$

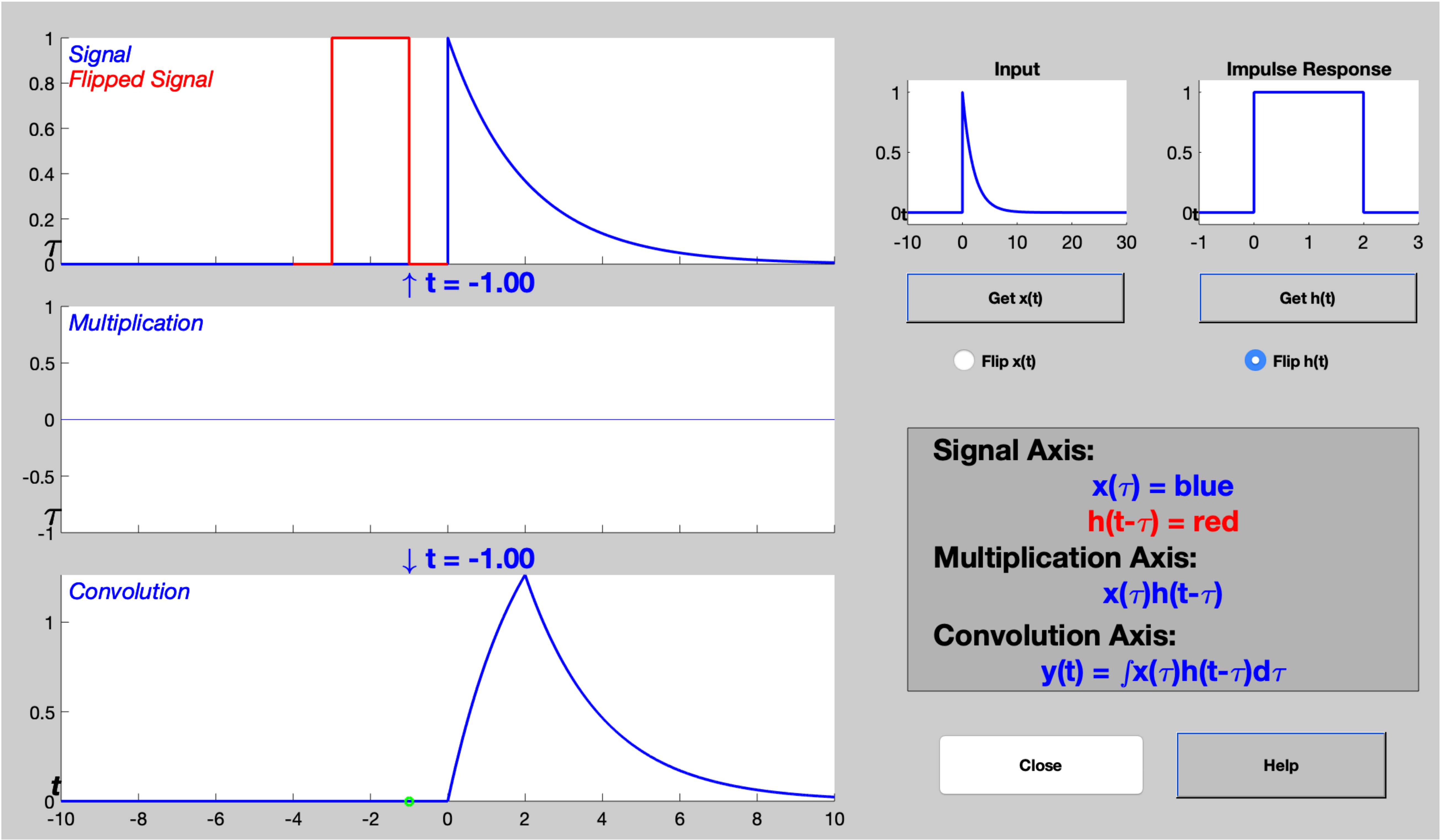
$$= \int_{\tau=-\infty}^{\infty} e^{-a\tau}u(\tau)(u(t-\tau) - u(t - \tau - t_0))d\tau \tag{8}$$

$$= \begin{cases} 0 & t \leq 0 \\ \int_{\tau=0}^{t} e^{-a\tau} d\tau & 0 < t \leq t_0 \\ \int_{\tau=t-t_0}^{t} e^{-a\tau} d\tau & t_0 < t < \infty \end{cases} \tag{9}$$

$$= \begin{cases} 0 & t \leq 0 \\ \frac{1}{a}(1 - e^{-at}) & 0 < t \leq t_0 \\ \frac{1}{a}(e^{-a(t-\tau)} - e^{-at})d\tau & t_0 < t < \infty \end{cases} \tag{10}$$

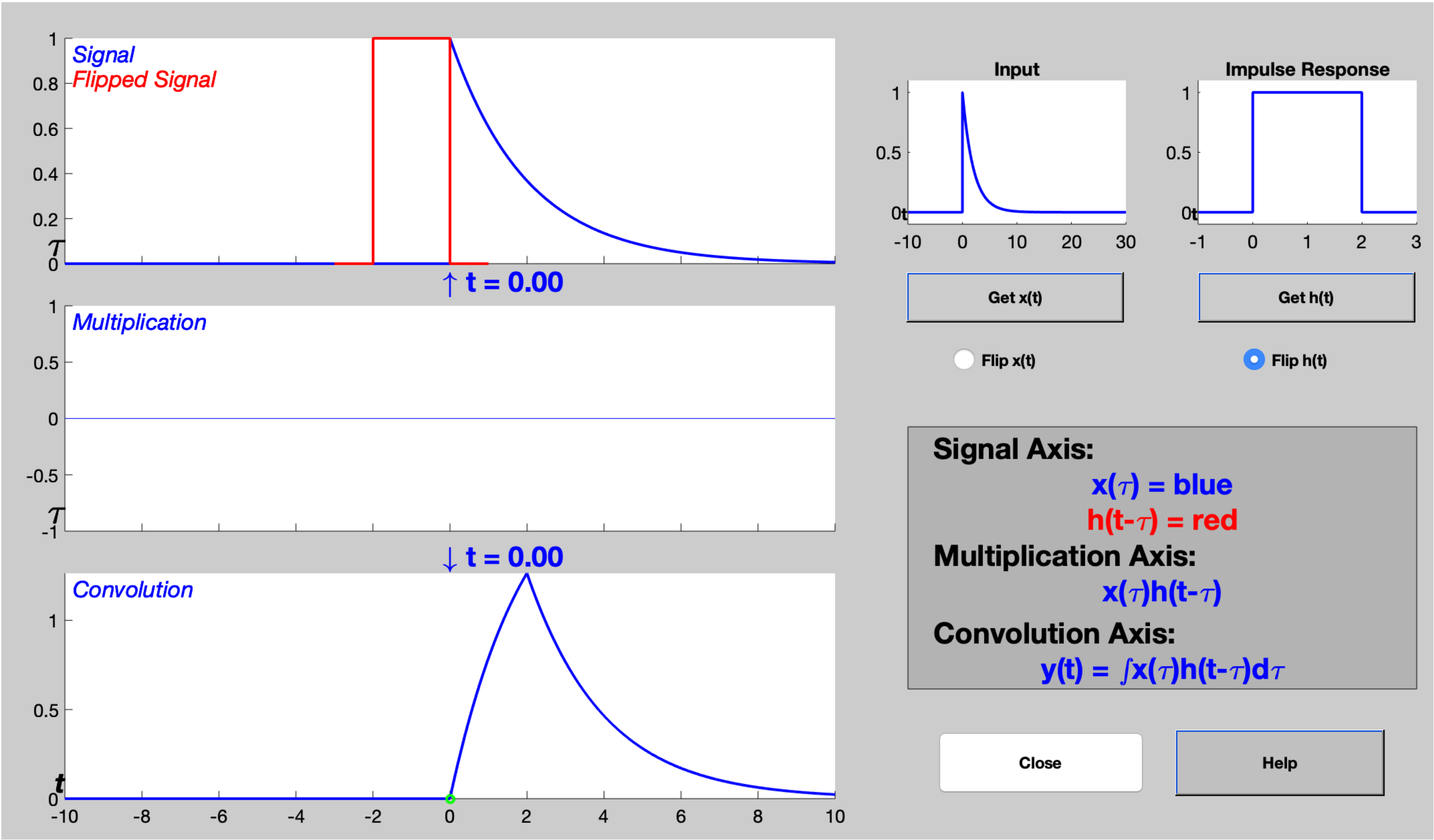
What's going on in pictures

DSP First @ Georgia Tech.



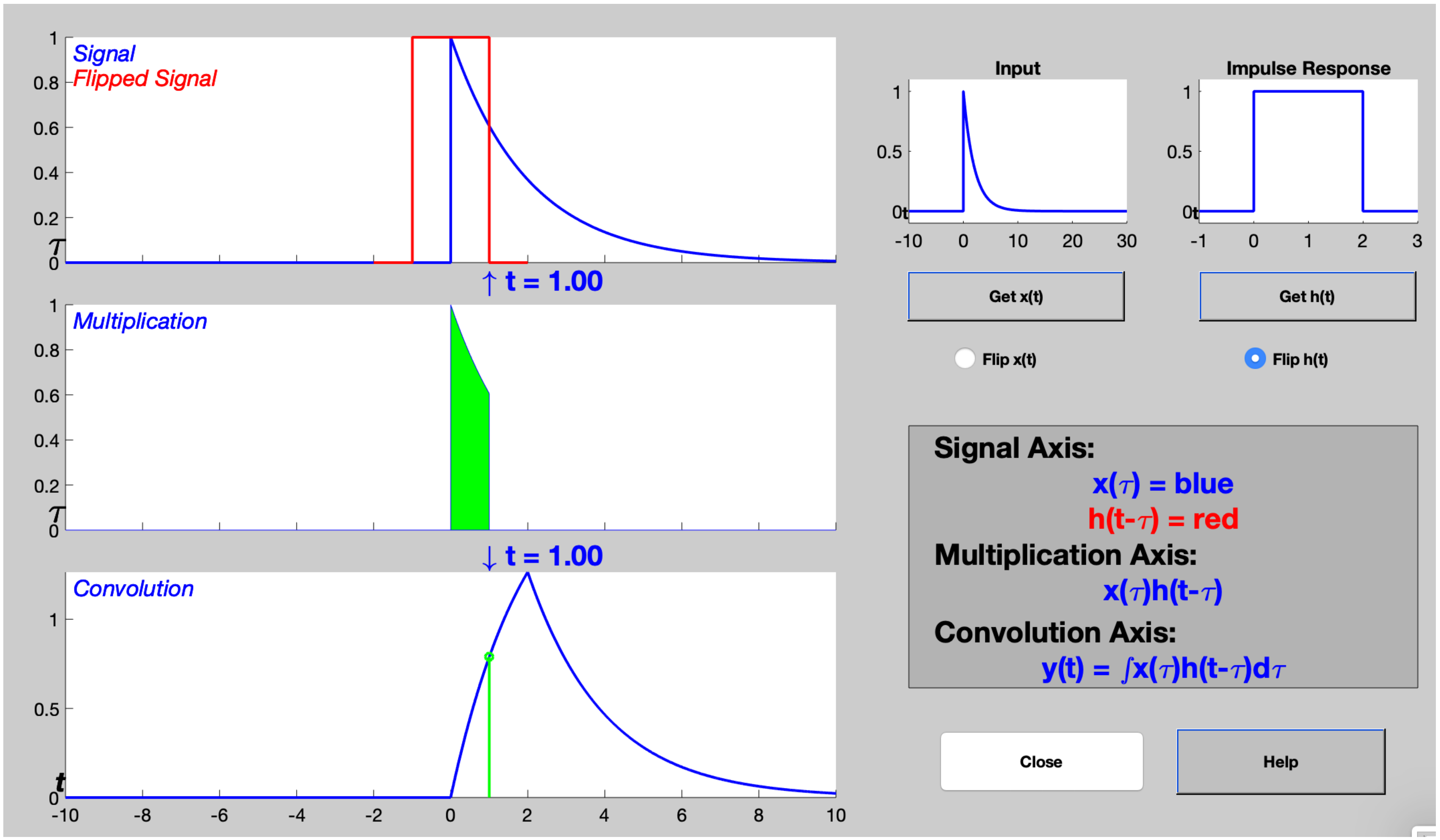
$$y(t) = 0 \qquad t \leq 0$$

What's going on in pictures



$$y(t) = 0 \qquad t \leq 0$$

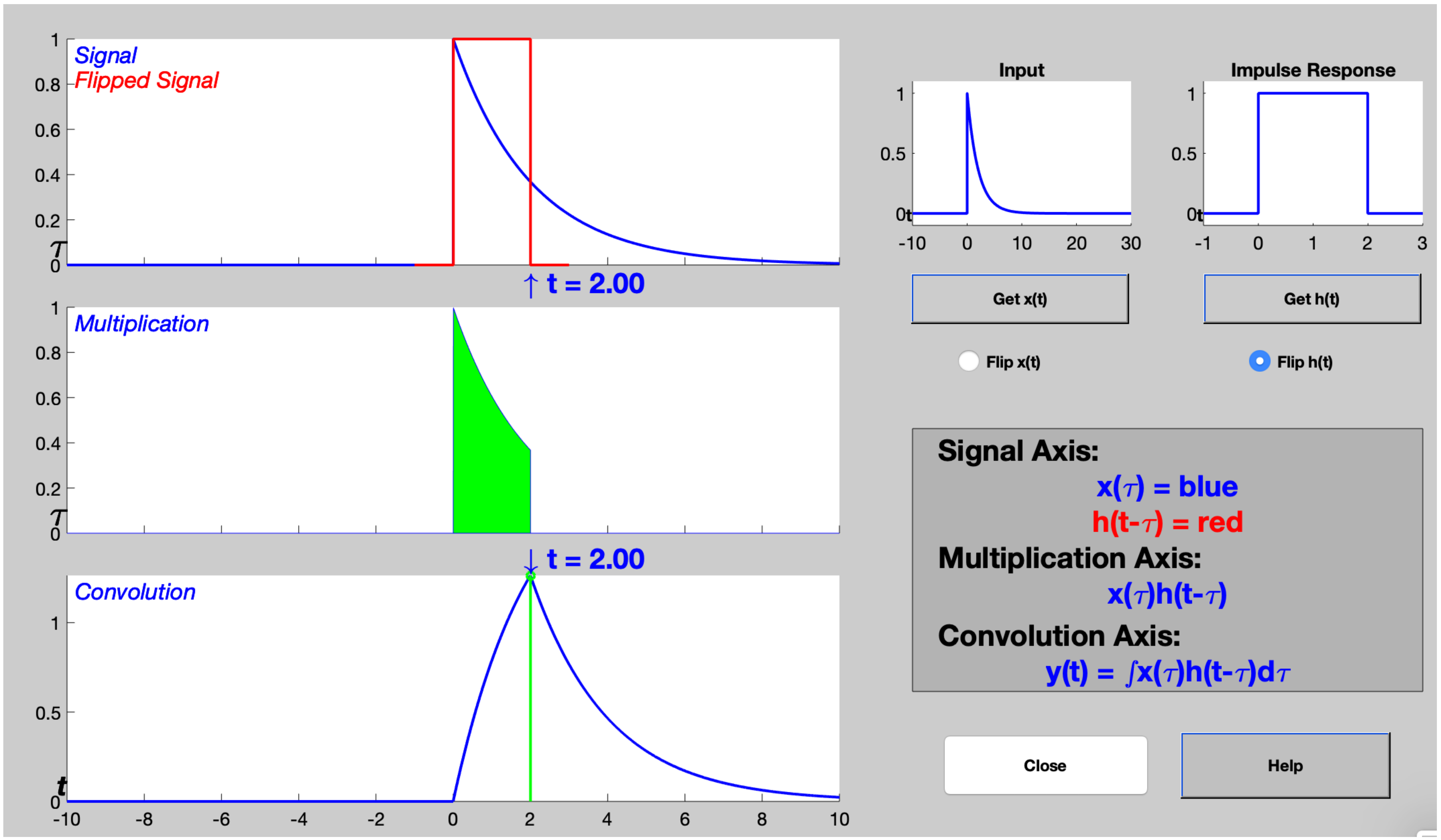
What's going on in pictures



$$y(t) = \frac{1}{a}(1 - e^{-at})$$

$0 < t \leq t_0$

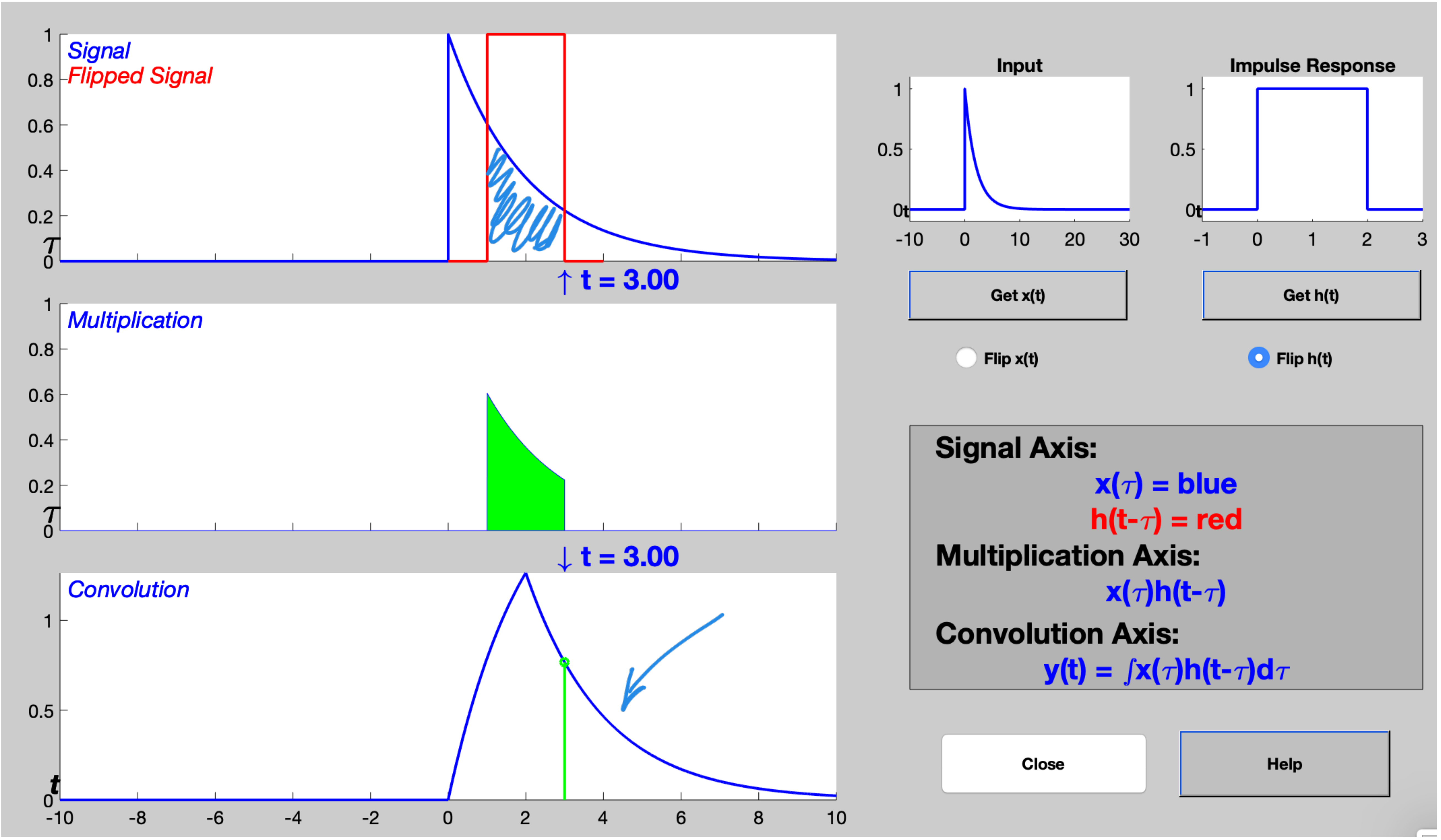
What's going on in pictures



$$y(t) = \frac{1}{a}(1 - e^{-at})$$

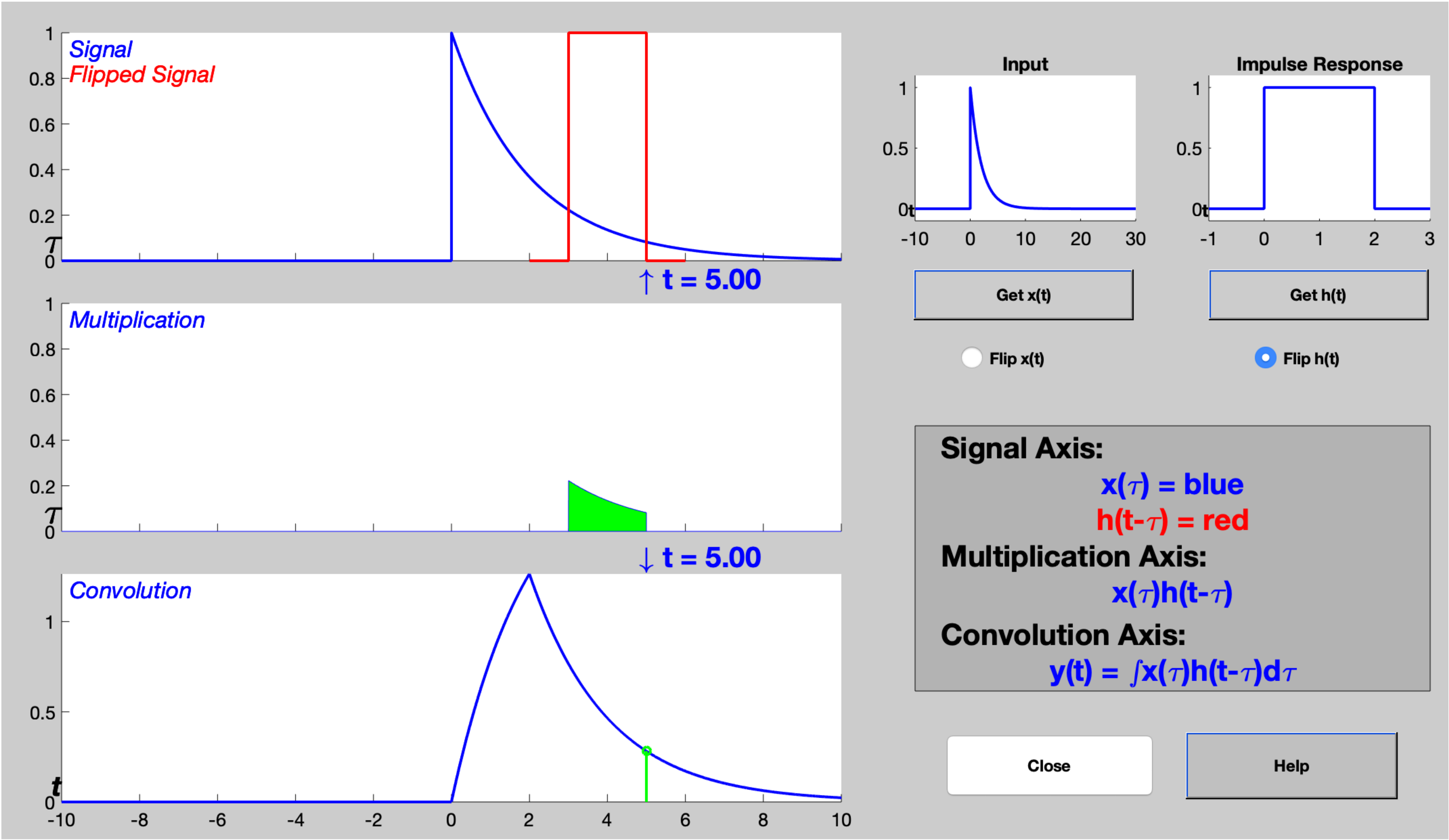
$0 < t \le t_0$

What's going on in pictures



$$y(t) = \frac{1}{a}(e^{-a(t-\tau)} - e^{-at})d\tau \qquad t_0 < t < \infty$$

What's going on in pictures



$$y(t) = \frac{1}{a}(e^{-a(t-\tau)} - e^{-at})d\tau \qquad t_0 < t < \infty$$

Another template

Suppose $x(t) = e^{-at}u(t)$ and $h(t) = e^{-bt}u(t)$ with $|a| < |b|$. Then

$$
\begin{aligned}
y(t) &= (x * h)(t) && (11) \\
&= \int_{\tau=\infty}^{\infty} e^{-a\tau}u(\tau)\underbrace{e^{-b(t-\tau)}u(t-\tau)}_{h(t-\tau)}d\tau && (12) \\
&= e^{-bt}\int_{0}^{t} e^{(b-a)\tau}d\tau && (13) \\
&= e^{-bt}\frac{1}{b-a}\left(e^{(b-a)t\cancel{-1}} - 1\right)u(t) && (14) \\
&= \frac{e^{-at} - e^{-bt}}{b-a}u(t) && (15)
\end{aligned}
$$

Compare this to the discrete-time analogue.

Using the new template

Problem

What is the output of an LTI system with impulse response $h(t) = 3e^{-2t}u(t)$ to an input $x(t) = 2e^{-(t-3)}u(t-3)$?

This is a total gain of 6 and a delay of 3 applied to the template:

$$\frac{e^{-t} - e^{-2t}}{2-1} u(t) * \underbrace{\delta(t-3)}_{\text{total delay}} * \underbrace{6}_{\text{total gain}} = \underbrace{6(e^{-(t-3)} - e^{-2(t-3)})u(t-3)}_{\text{plug in to template}}. \quad (16)$$

Try it yourself

Problem

Calculate the output of the system $h(t)$ with input $x(t)$:

$$x(t) = e^{-2(t-5)}u(t), h(t) = -2u(t) \tag{17}$$

$$x(t) = u(t+2) - u(t-3), h(t) = -e^{-4(t-1)}u(t) \tag{18}$$

$$x(t) = 2e^{-3t+2}u(t-2), h(t) = e^{2(t-1)}u(t) \tag{19}$$

Make up different $x(t)$ and $h(t)$, convolve them, and use those as a "template" to solve problems.