Linear Systems and Signals
CT Convolution Theorem

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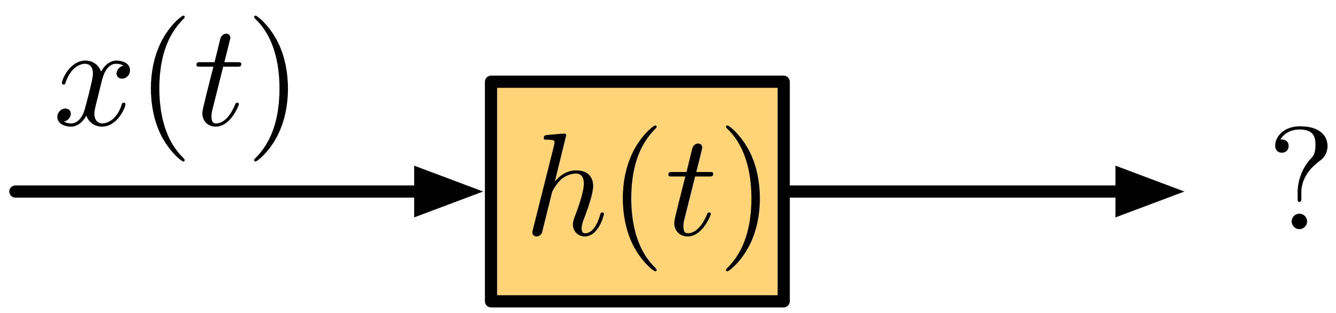
Learning objectives

The learning objective for this video is:

- explain why the output of a CT LTI system is the convolution of the input with the impulse response

Convolution in CT

$\delta(t) \longrightarrow \boxed{\mathcal{H}} \longrightarrow h(t)$



We saw that for DT LTI systems, the output is the discrete convolution of the input and the impulse response. In CT we also have an impulse response using the *Dirac delta* function and a convolution operation:

$$(x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau. \tag{1}$$

def of convolution

Is the convolution the output for CT LTI systems?

CT impulse response

$$\delta(t) \longrightarrow \boxed{\mathcal{H}} \longrightarrow h(t)$$

Define the *impulse response* of the LTI system $\mathcal{H}$ as the output signal with input $\delta(t)$. Since the system is LTI,

$$\delta(t) \xrightarrow{\mathcal{H}} h(t) \tag{2}$$

time invariant

delay

$$\delta(t-\tau) \xrightarrow{\mathcal{H}} h(t-\tau) \tag{3}$$

linearity

$$x(\tau)\delta(t-\tau) \xrightarrow{\mathcal{H}} x(\tau)h(t-\tau) \tag{4}$$

Now instead of summing as we would in DT, we *integrate* both sides over τ.

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \xrightarrow{\mathcal{H}} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{5}$$

CT convolution

The convolution theorem

Now we apply the *sifting property* of the delta function:

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \xrightarrow{\mathcal{H}} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \quad (6)$$

$$x(t) \xrightarrow{\mathcal{H}} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \quad (7)$$

CT convolution

Theorem

Let $\mathcal{H}$ be a CT linear time-invariant (LTI) system with impulse response $h(t)$. Then the output $y(t)$ to an input signal $x(t)$ is the continuous convolution of $x(t)$ and $h(t)$:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau. \quad (8)$$

The impulse response contains everything

$h(t)$ is a signal

signal properties of $h(t)$

$\Updownarrow$

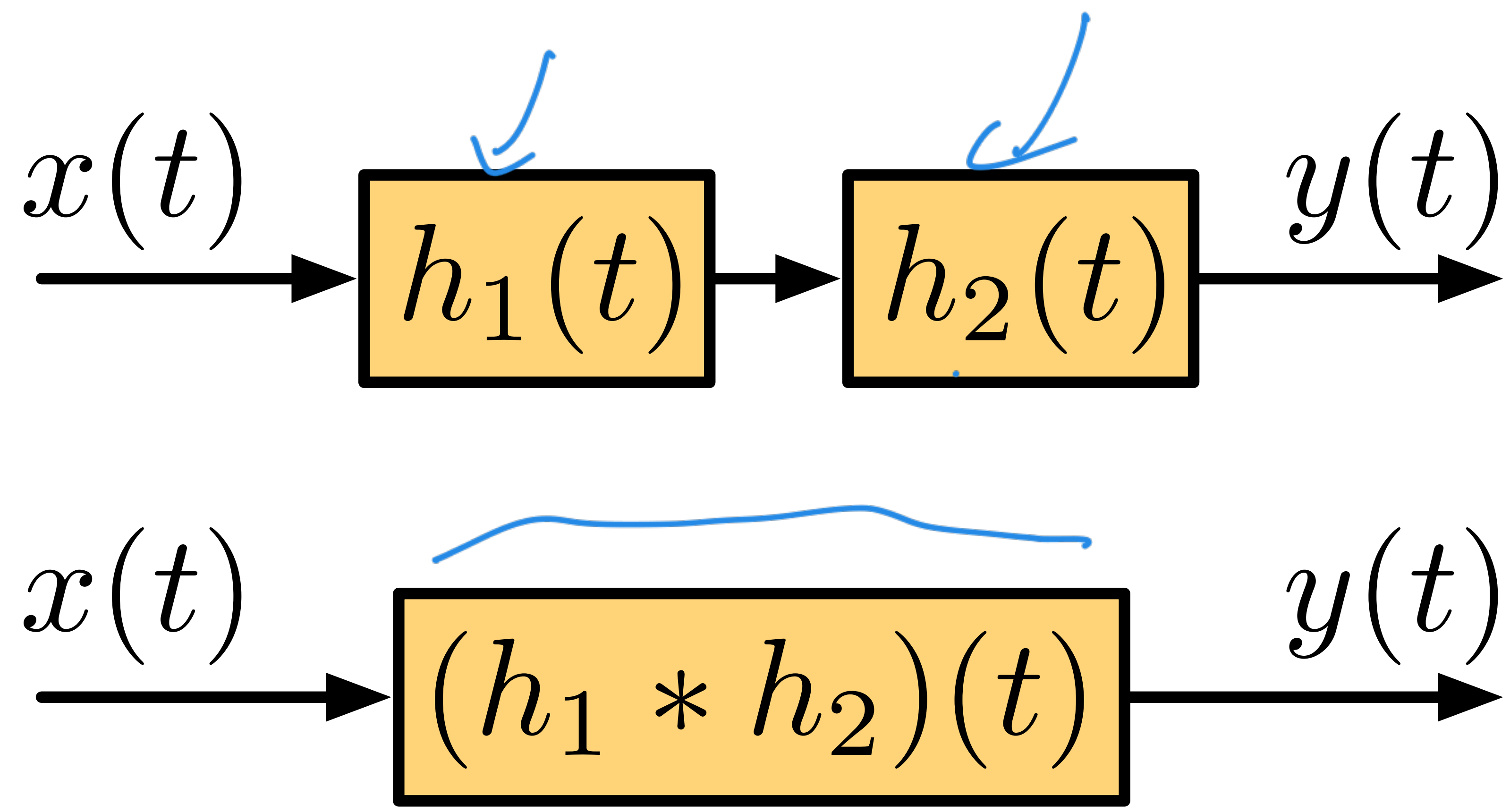
system properties $\mathcal{H}$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau. \tag{9}$$

What this means is that *the impulse response contains everything you need to know about the system*:

- to calculate the output of a system you just need the input and impulse response
- system properties (stability, causality, etc.) are properties of the impulse response

Convolution properties



same output

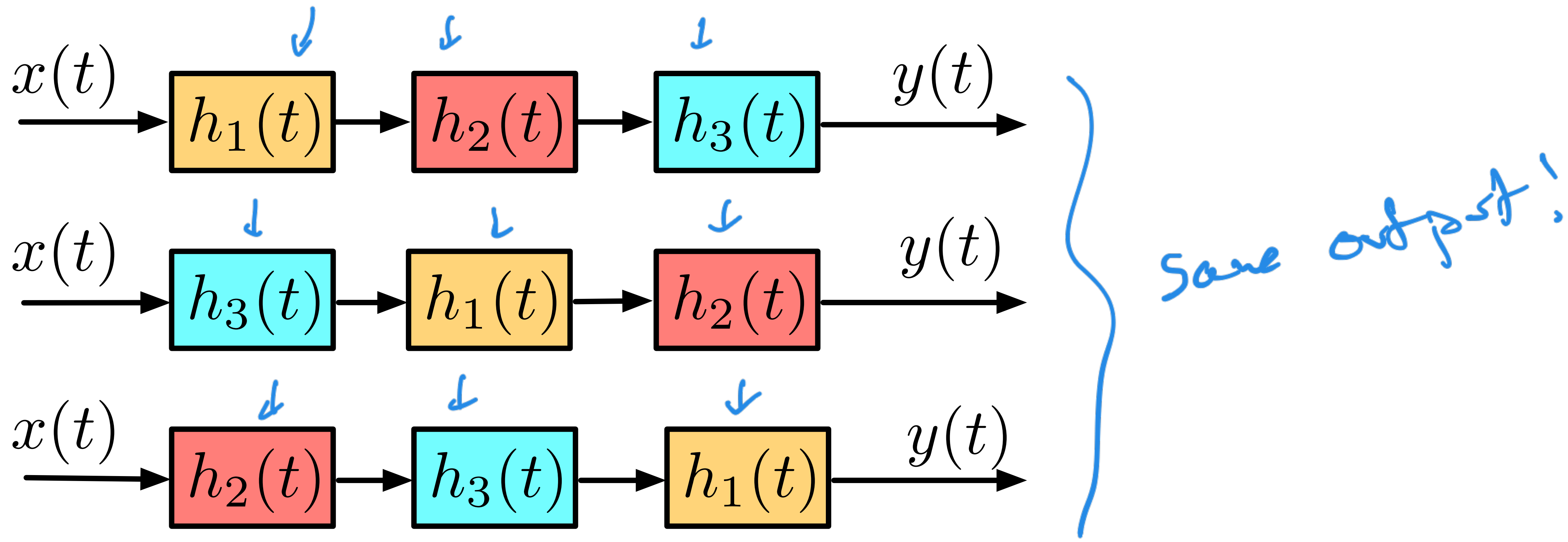
CT convolution is commutative (just like DT):

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = (x*h)(t) = (h*x)(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \tag{10}$$

and associative:

$$((h_1 * h_2) * x)(t) = ((h_2 * (h_1 * x))(t). \tag{11}$$

Order-invariance



This means that we can apply a chain of LTI systems in whatever order we want. Why should we choose one over the other?

- Cost
- Power consumption
- Interference or other implementation issues

Calculating the convolution

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\underbrace{h(t-\tau)}_{\text{delayed copy of } h(t)}d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \qquad (12)$$

We can see that the output is a *mixture* of copies of the impulse response starting at different times τ. Unfortunately, we cannot use the superposition trick like in DT very often. So we have two main approaches:

1. Graphical: use the "flip-and-slide" view to compute the integral at each time t.
2. Algebraic: write out the convolution formula and use standard integration tricks or reuse known formulas.