Linear Systems and Signals

Computing the convolution with flip and slide

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Learning objectives

The learning objective for this section is:

- calculate simple convolutions using the flip and slide method

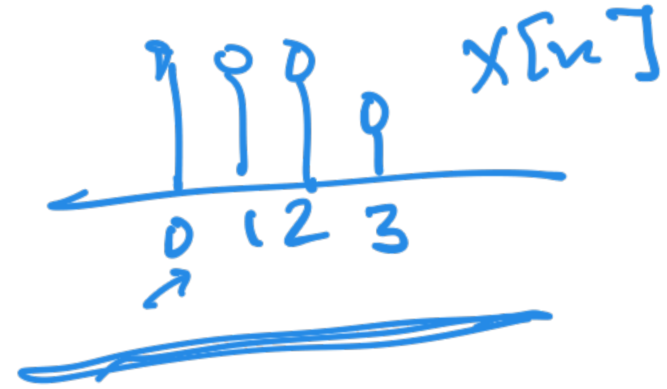
Impulse response and convolution

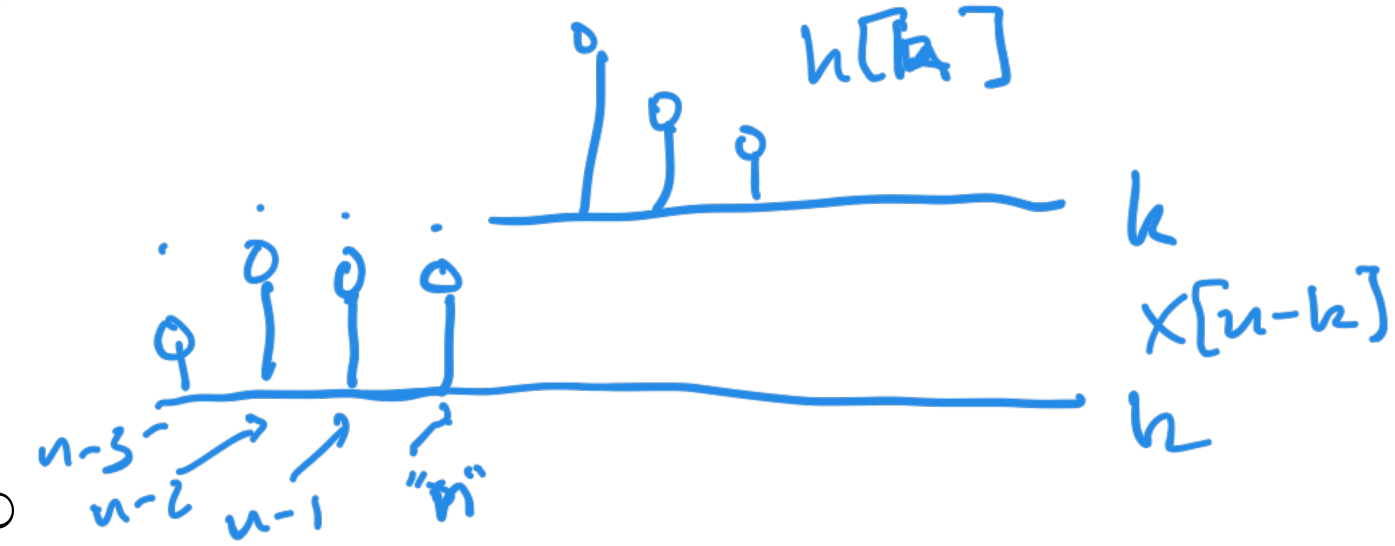
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \qquad (1)$$

For "simple" signals and filters with "simple" impulse responses we can sometimes just compute the convolution directly:

1. Interpret the output as scaled and shifted copies of the impulse response $h[n]$ or as scaled and shifted copies of $x[n]$.
2. Use the "flip-and-slide" view to compute the product $h[k]x[n-k]$ for each n and add it up to get $y[n]$.
3. Write out the convolution formula and use formulas such as power series to simplify the expression for the output.

The “flip and slide” method





Look at the formula

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \qquad (2)$$

For the output at time n, we can interpret this as:

- “flip” the signal x to make it enter the system in the correct order
- shift it forward by n time steps
- multiply x and h where they overlap and add it up (like a dot product)

Example

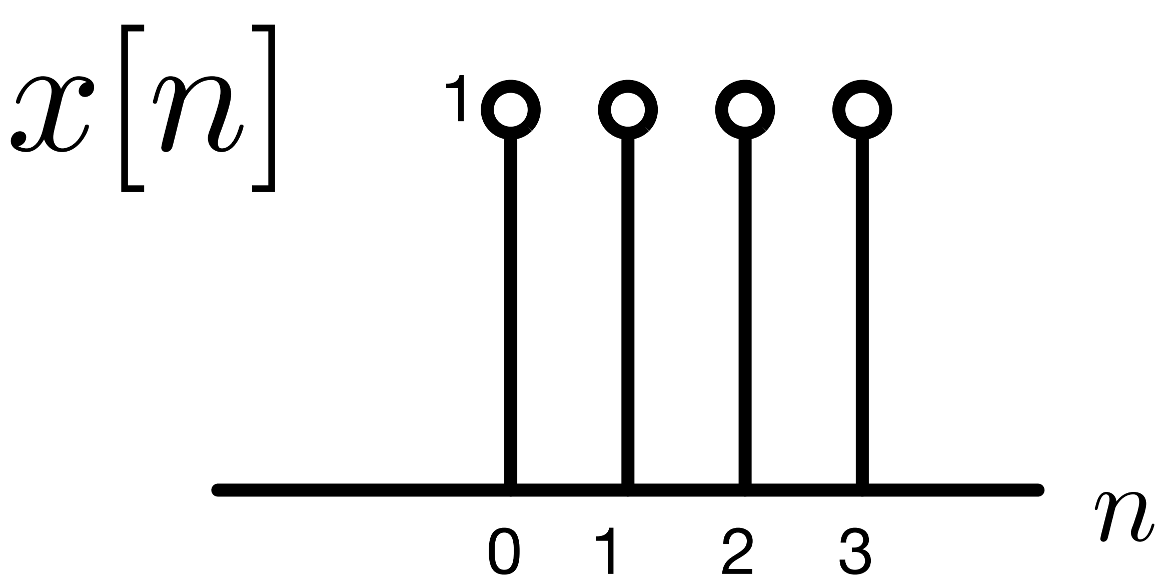
Let's revisit our previous example: find $y[n] = (x * h)[n]$ when

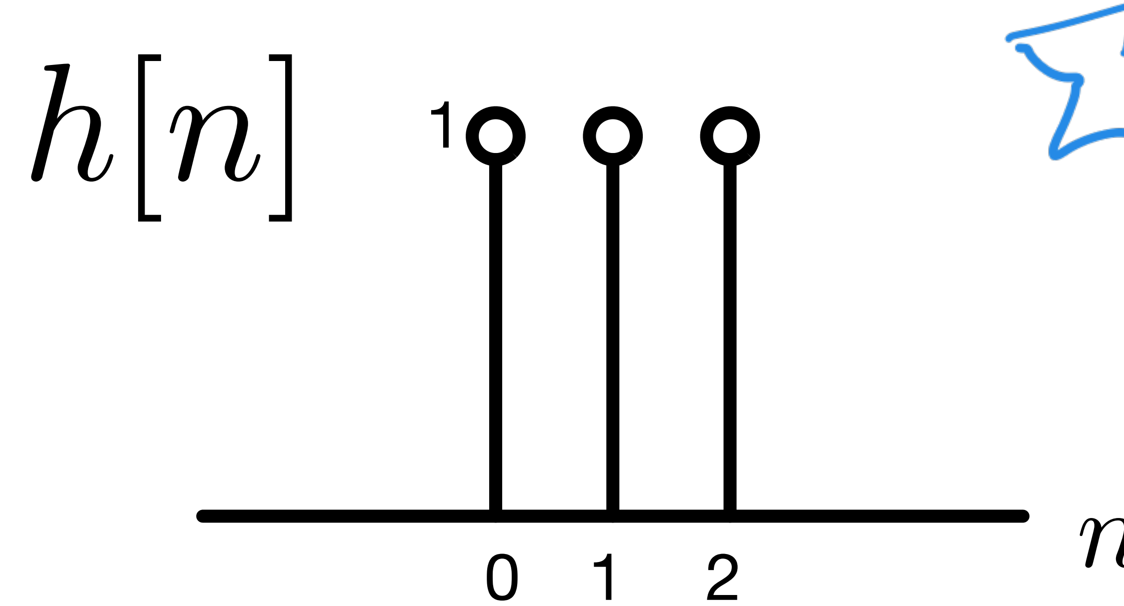
$$x[n] = u[n] - u[n-4]$$
$$h[n] = u[n] - u[n-3]$$

Step 0: draw a picture and rewrite the signals if needed:

$\sum h[k]x[n-k]$

k-axis

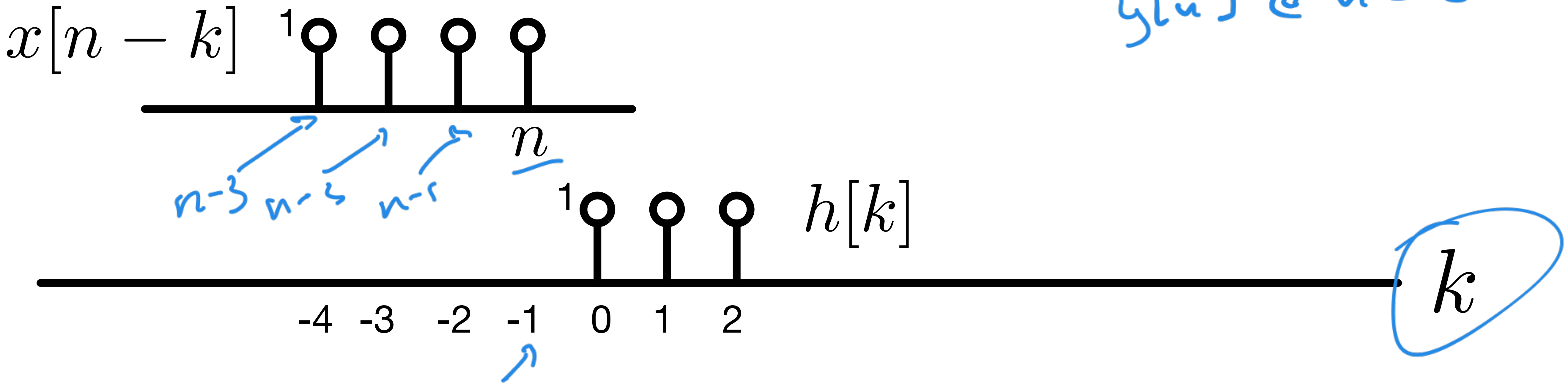




$$x[n] = \begin{cases} 1 & 0 \le n \le 3 \\ 0 & \text{otherwise} \end{cases} \qquad h[n] = \begin{cases} 1 & 0 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Visual/graphical approach

y[n] @ n = 0

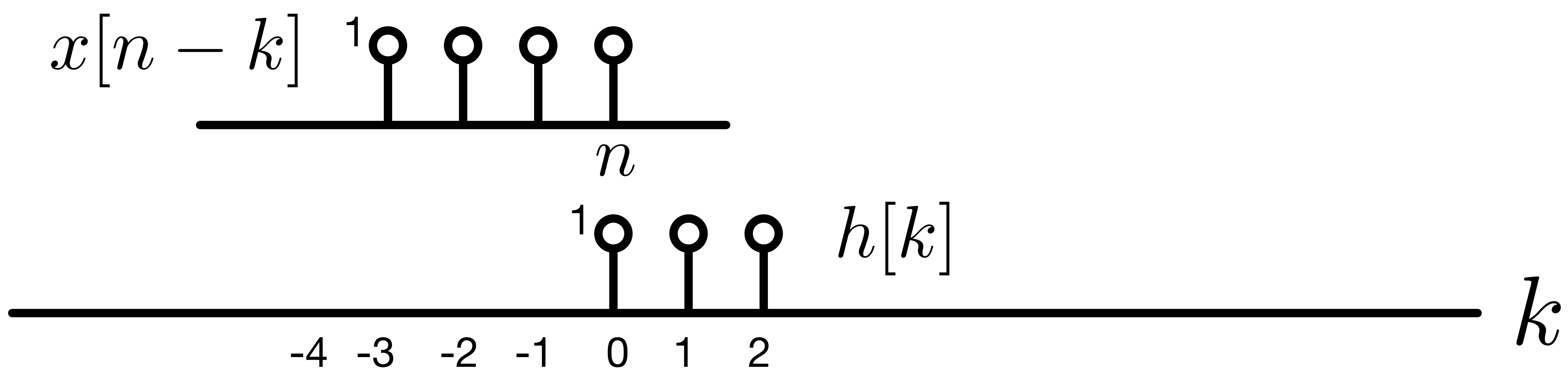


$$n = -1$$

1. "Flip" $x[k]$ to $x[-k]$ and look at different shifts by n.
2. Multiple shifted signal $x[n-k]$ and impulse response $h[k]$.
3. Add up to get the $y[n]$ value.

$$y[n] =$$

Visual/graphical approach

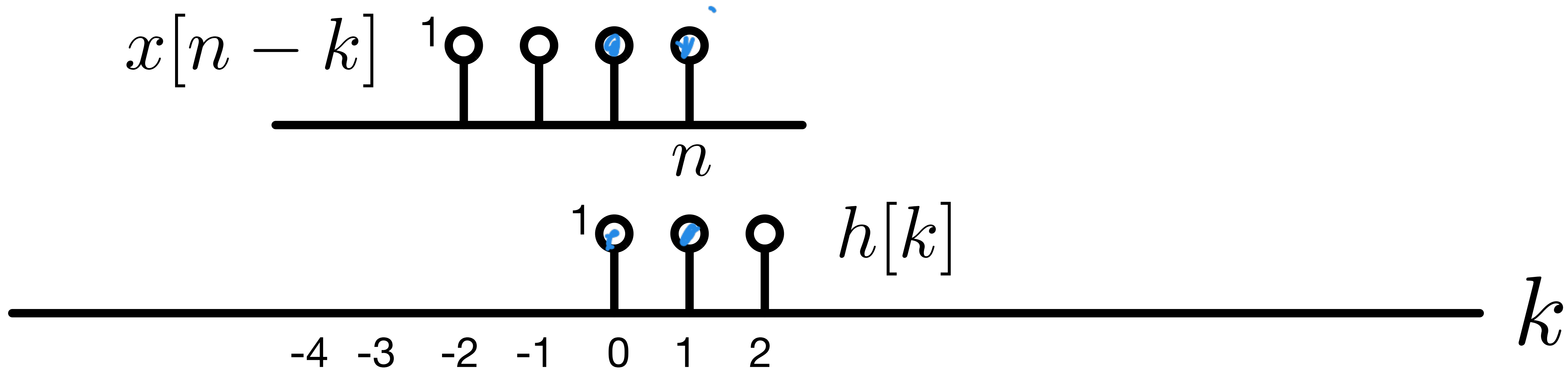


$$n = 0$$

1. "Flip" $x[k]$ to $x[-k]$ and look at different shifts by n.
2. Multiple shifted signal $x[n-k]$ and impulse response $h[k]$.
3. Add up to get the $y[n]$ value.

$$y[n] = \delta[n]$$

Visual/graphical approach

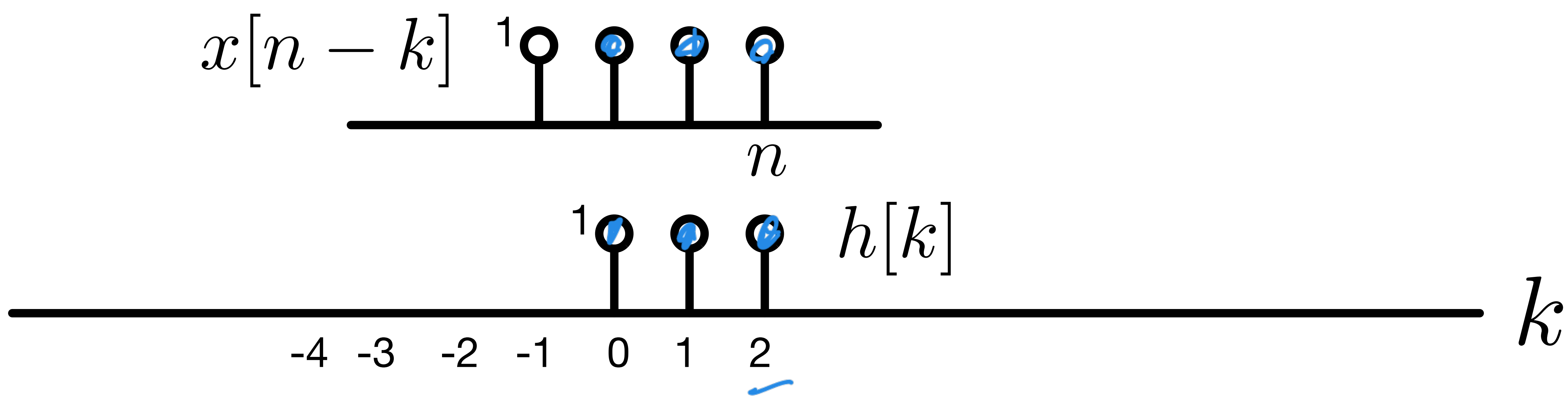


$$n = 1$$

1. "Flip" $x[k]$ to $x[-k]$ and look at different shifts by n.
2. Multiple shifted signal $x[n-k]$ and impulse response $h[k]$.
3. Add up to get the $y[n]$ value.

$$y[n] = \delta[n] + 2\delta[n-1]$$

Visual/graphical approach

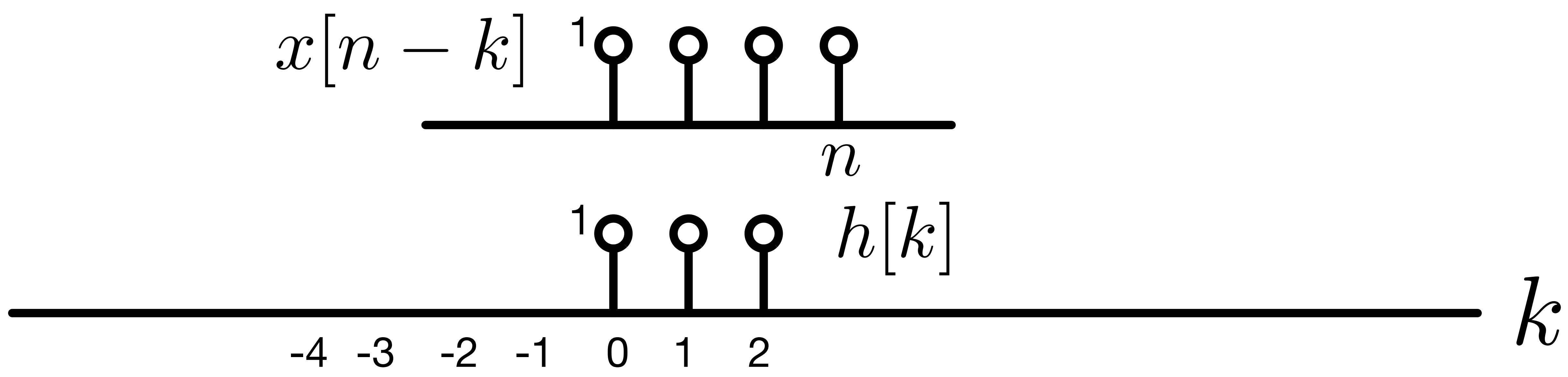


$$n = 2$$

1. "Flip" $x[k]$ to $x[-k]$ and look at different shifts by n.
2. Multiple shifted signal $x[n-k]$ and impulse response $h[k]$.
3. Add up to get the $y[n]$ value.

$$y[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2]$$

Visual/graphical approach

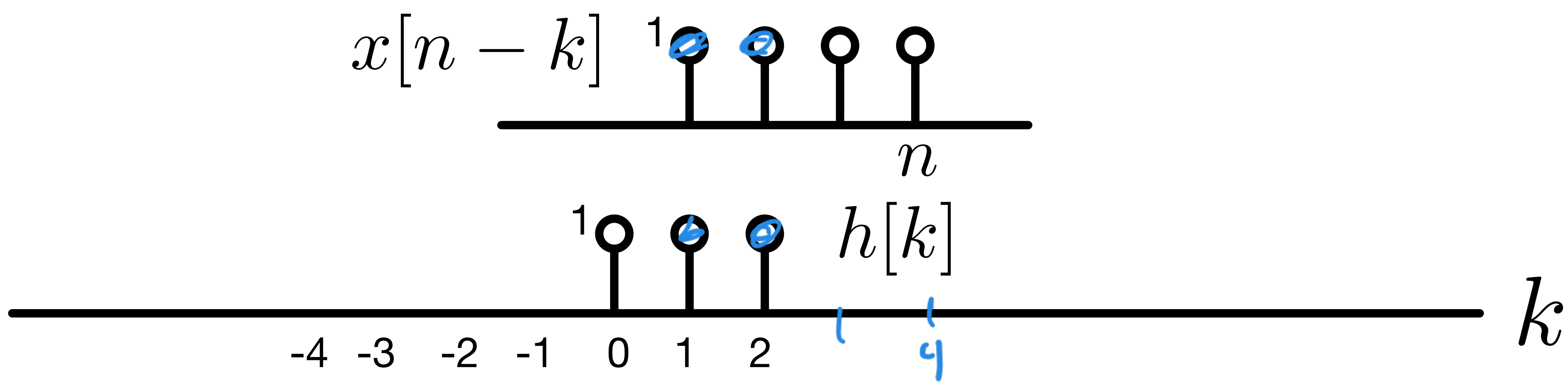


$$n = 3$$

1. "Flip" $x[k]$ to $x[-k]$ and look at different shifts by n.
2. Multiple shifted signal $x[n-k]$ and impulse response $h[k]$.
3. Add up to get the $y[n]$ value.

$$y[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 3\delta[n-3]$$

Visual/graphical approach

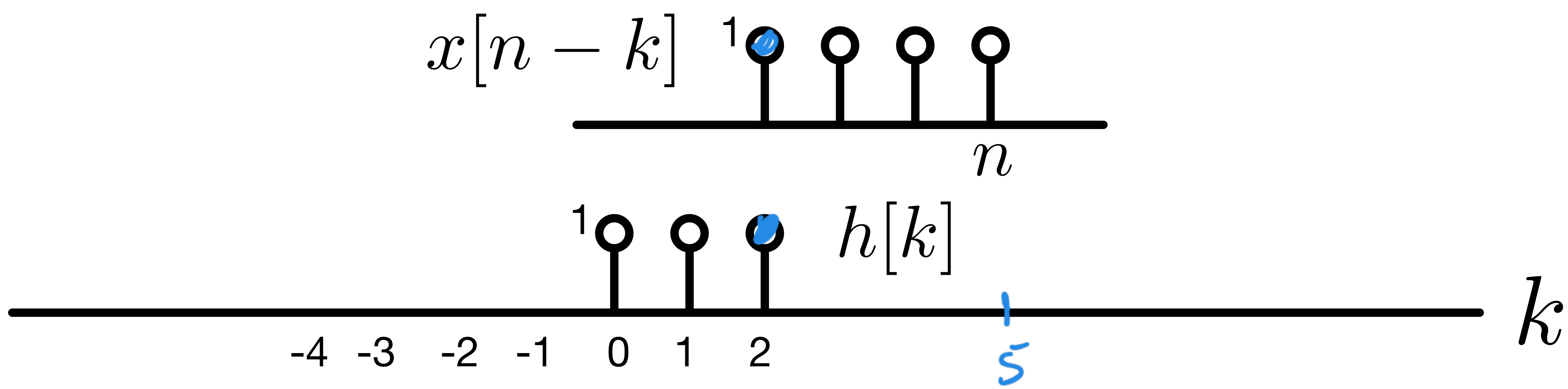


$n = 4$

1. "Flip" $x[k]$ to $x[-k]$ and look at different shifts by n.
2. Multiple shifted signal $x[n-k]$ and impulse response $h[k]$.
3. Add up to get the $y[n]$ value.

$$y[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 3\delta[n-3] \\ + 2\delta[n-4]$$

Visual/graphical approach

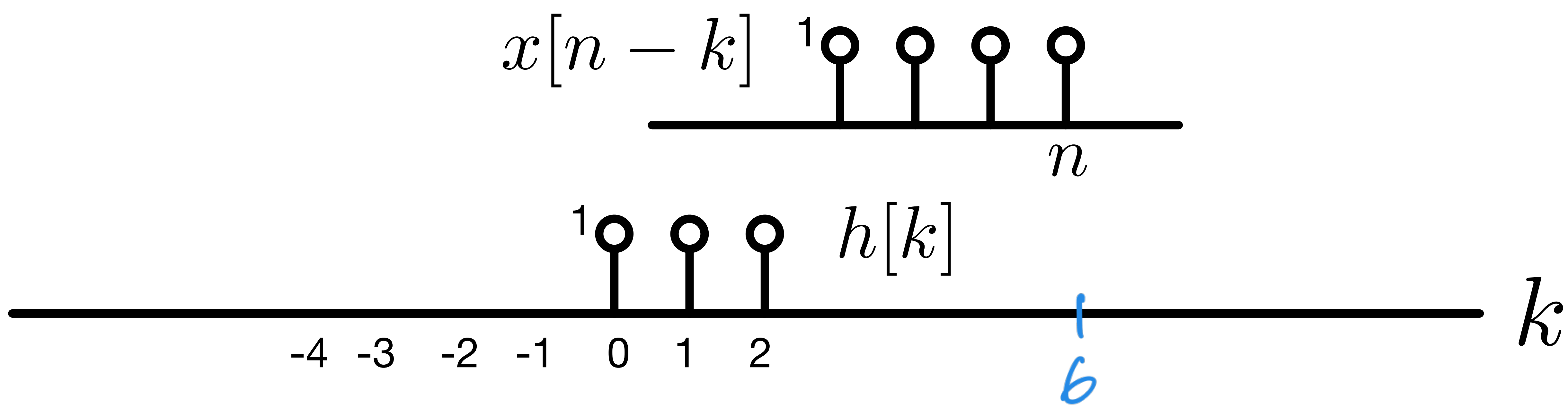


$$n = 5$$

1. "Flip" $x[k]$ to $x[-k]$ and look at different shifts by n.
2. Multiple shifted signal $x[n-k]$ and impulse response $h[k]$.
3. Add up to get the $y[n]$ value.

$$y[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 3\delta[n-3] \\ + 2\delta[n-4] + \delta[n-5]$$

Visual/graphical approach



$$n = 6$$

1. "Flip" $x[k]$ to $x[-k]$ and look at different shifts by n.
2. Multiple shifted signal $x[n-k]$ and impulse response $h[k]$.
3. Add up to get the $y[n]$ value.

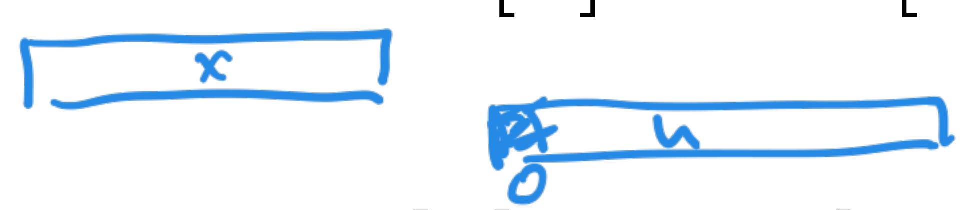
$$y[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 3\delta[n-3]$$
$$+ 2\delta[n-4] + \delta[n-5]$$

The 5-phase approach

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \tag{3}$$

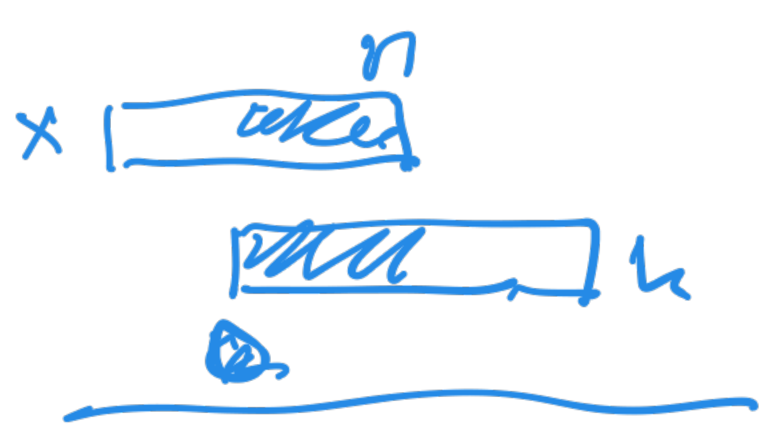
Suppose $x[n]$ is finite-length from 0 to N, $h[n]$ is finite-length from 0 to M and $N < M$.

1. **Phase 1**: $n < 0$: neither $h[k]$ nor $x[n-k]$ overlap. Then $y[n] = 0$.



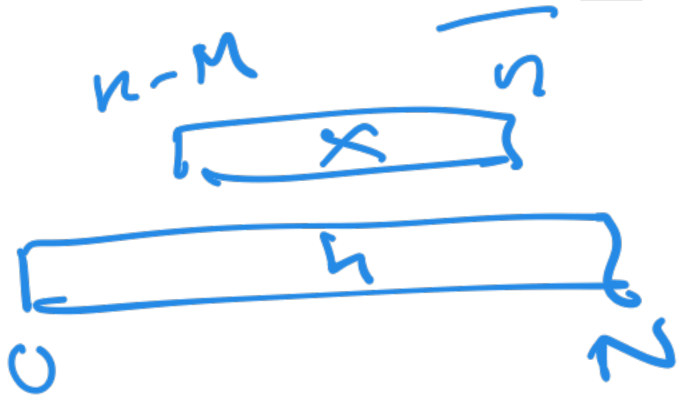
2. **Phase 2**: $0 \le n < M$: $h[k]$ and $x[n-k]$ partially overlap. Then

$$y[n] = \sum_{k=0}^{n} \boxed{h[k]x[n-k]}. \tag{4}$$



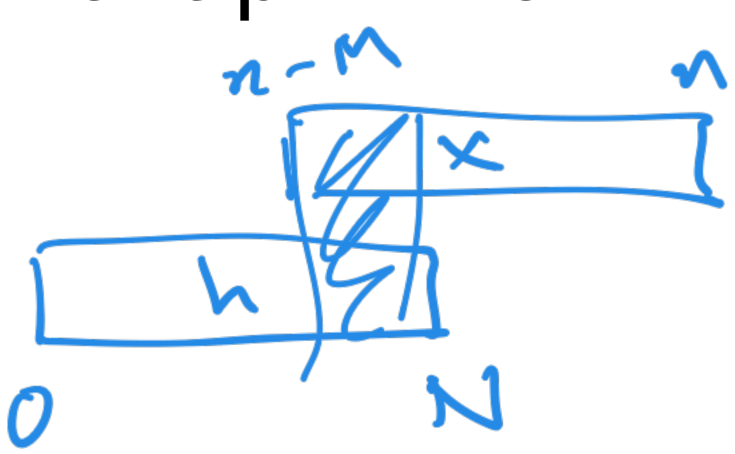
The 5-phase approach, continued

3 **Phase 3**: $M \le n < N$: $h[k]$ and $x[n-k]$ fully overlap. Then



$$y[n] = \sum_{k=n-M}^{n} \boxed{h[k]x[n-k].} \qquad (5)$$

4 **Phase 4**: $N \le n < N + M + 1$: $h[k]$ and $x[n-k]$ partially overlap. Then



$$y[n] = \sum_{k=n-M}^{N} h[k]x[n-k]. \qquad (6)$$

$\le n < \infty$

5 **Phase 5**: $N + M + 1 \cancel{\le} \infty$: neither $h[k]$ nor $x[n-k]$ overlap. Then $y[n] = 0$.

$N+M+1 \le n < \infty$

len(x) + len(h) - 1

A useful MATLAB demo

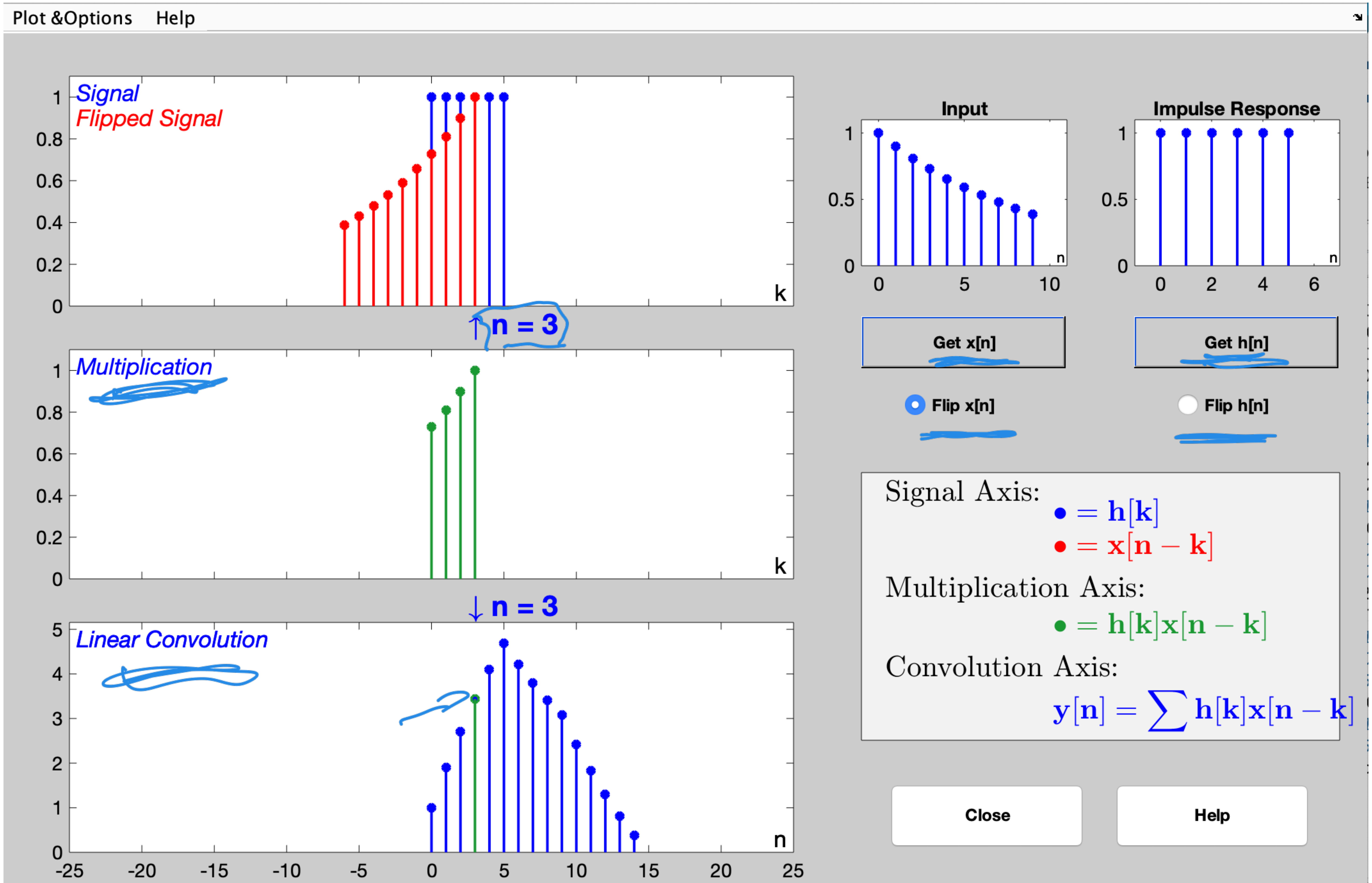
It helps to be able to play around with the graphical approach yourself. The DSP First program at Georgia Tech has made a nice tool to help visualize this flip and slide method:

`http://dspfirst.gatech.edu/matlab/#dconvdemo`

You have to download it and run it in MATLAB.

dconvdemo.m

Screenshot of the demo



Try some yourself

Problem

Find the convolution from the following input-output relations:

$$h[n] = u[n] - u[n-7], x[n] = u[n] - u[n-6]$$
$$h[n] = u[n] - u[n-5], x[n] = -\delta[n] + 2\delta[n-2]$$
$$h[n] = n(u[n] - u[n-3]), x[n] = 2\delta[n] - 2\delta[n-1]$$

Make up a few on your own!