Linear Systems and Signals
CT flip-and-slide example

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Learning objectives

The learning objectives for this section are:

- manually compute convolutions in the time domain using the 5-phase method
- interpret the output of the signal

An example application

Suppose you are measuring a radio transmission $s(t)$ which has some interference at a particular frequency f_0. To cancel it out you could try to implement a filter that integrates its input over a fixed window of time. What should you expect to see at the output?

- Nothing should happen until the receiver gets the signal
- There should be transients from the cosine starting
- There will be some steady-state behavior
- There will be more transients from the cosine ending
- Eventually the signal should die down

How can we understand this mathematically? These are the 5 phases of our convolution.

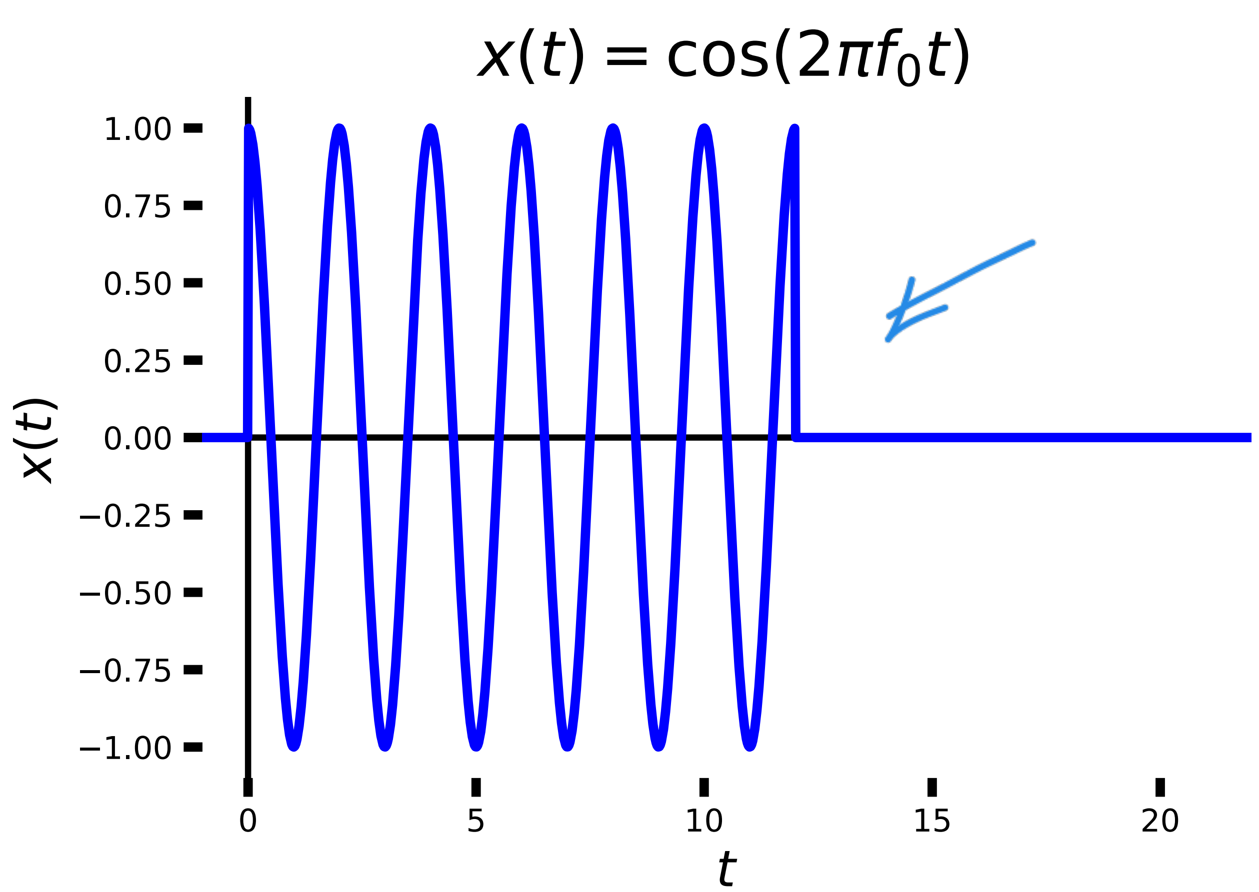
A cosine and a box

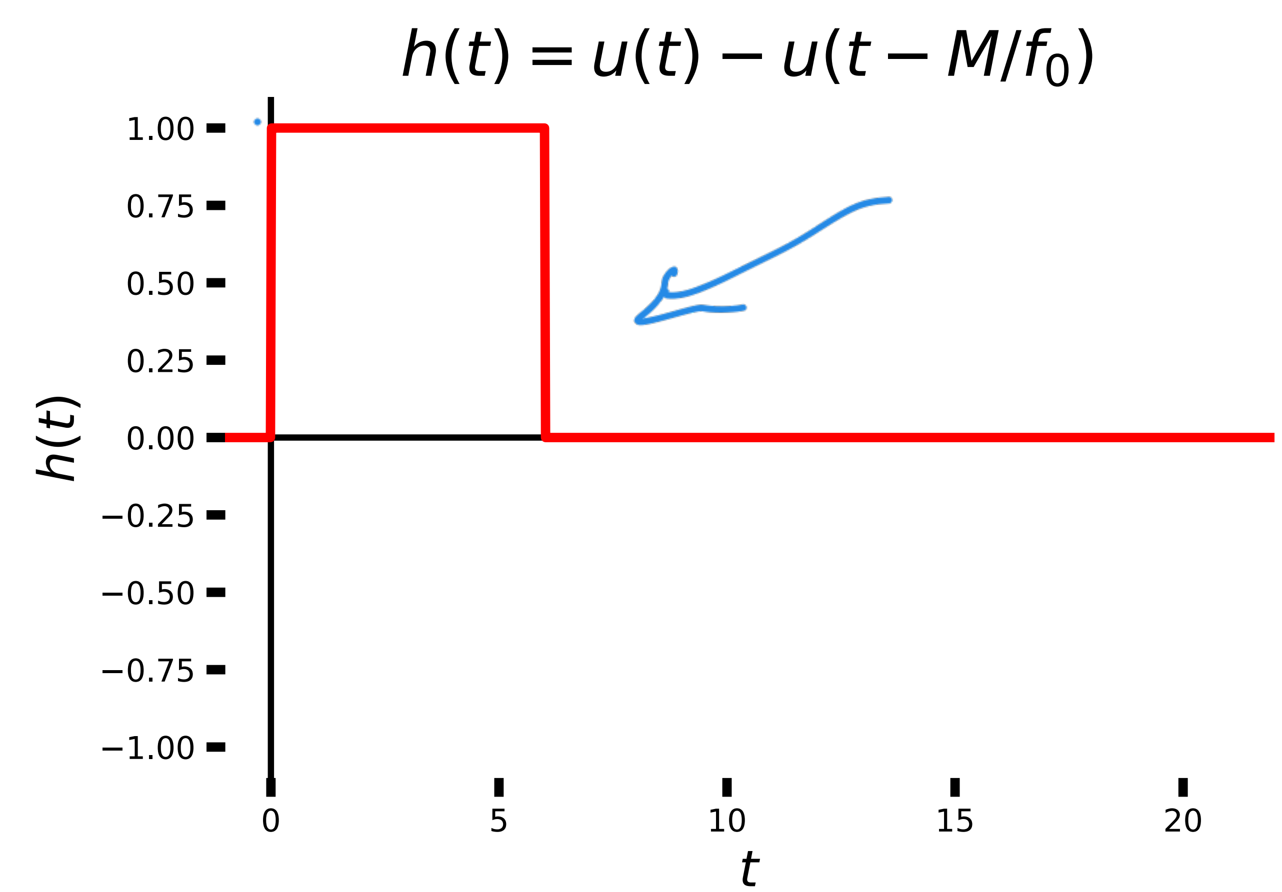
The actual signal is $s(t) + x(t)$ where $s(t)$ is the legitimate signal. Let's look at the interference signal $x(t)$. Suppose M, K are positive integers with $M < K$ and

$$x(t) = \underline{\cos(2\pi f_0 t)}(\overset{0}{u(t)} - \overset{K/f_0}{u(t - K/f_0)}) \tag{1}$$

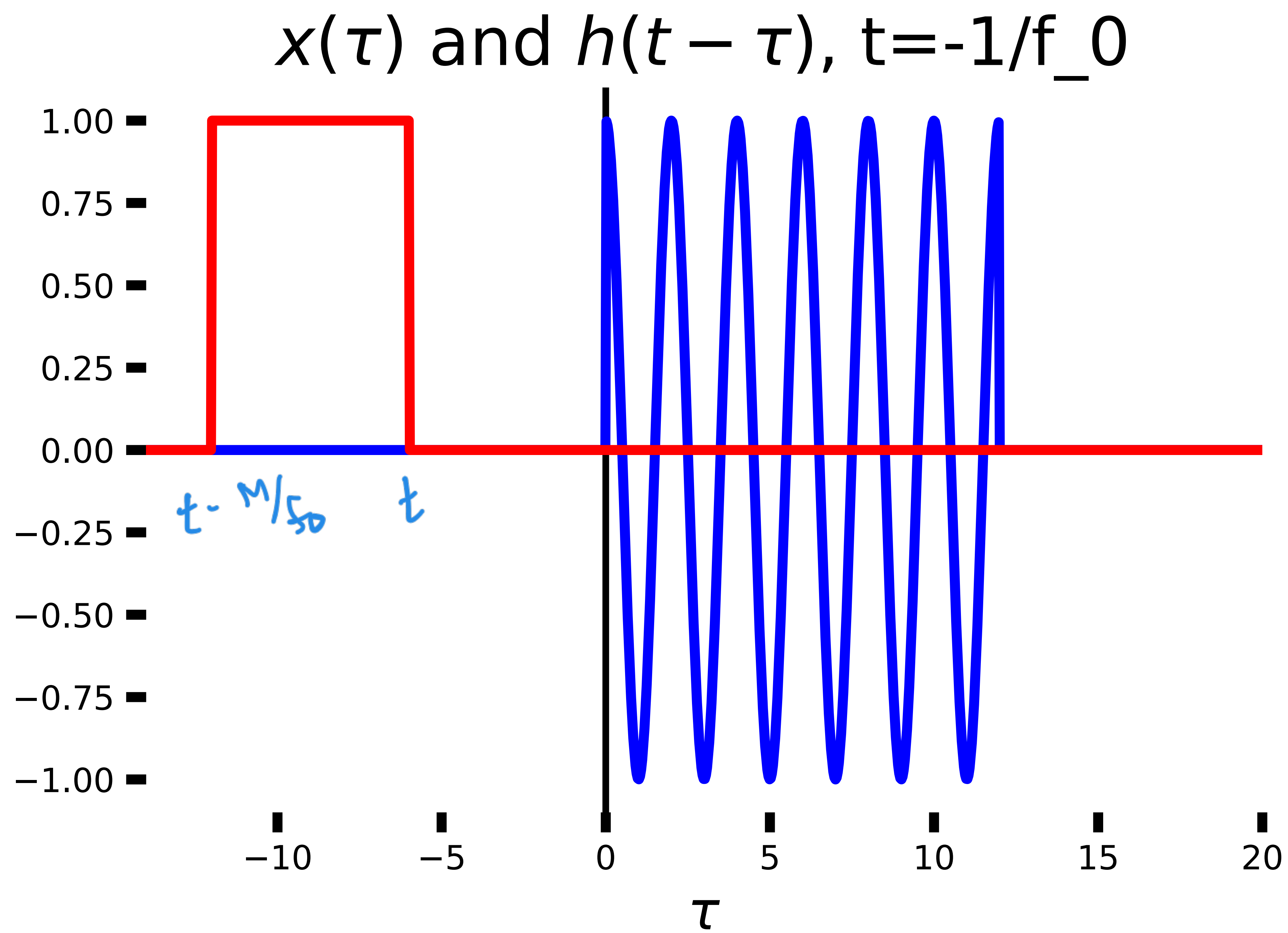
$$h(t) = \underline{u(t) - u(t - M/f_0)} \tag{2}$$

The convolution $(h * x)(t)$ will show the effect on the interference. Where to start? Step 0 is always to *draw a picture*:



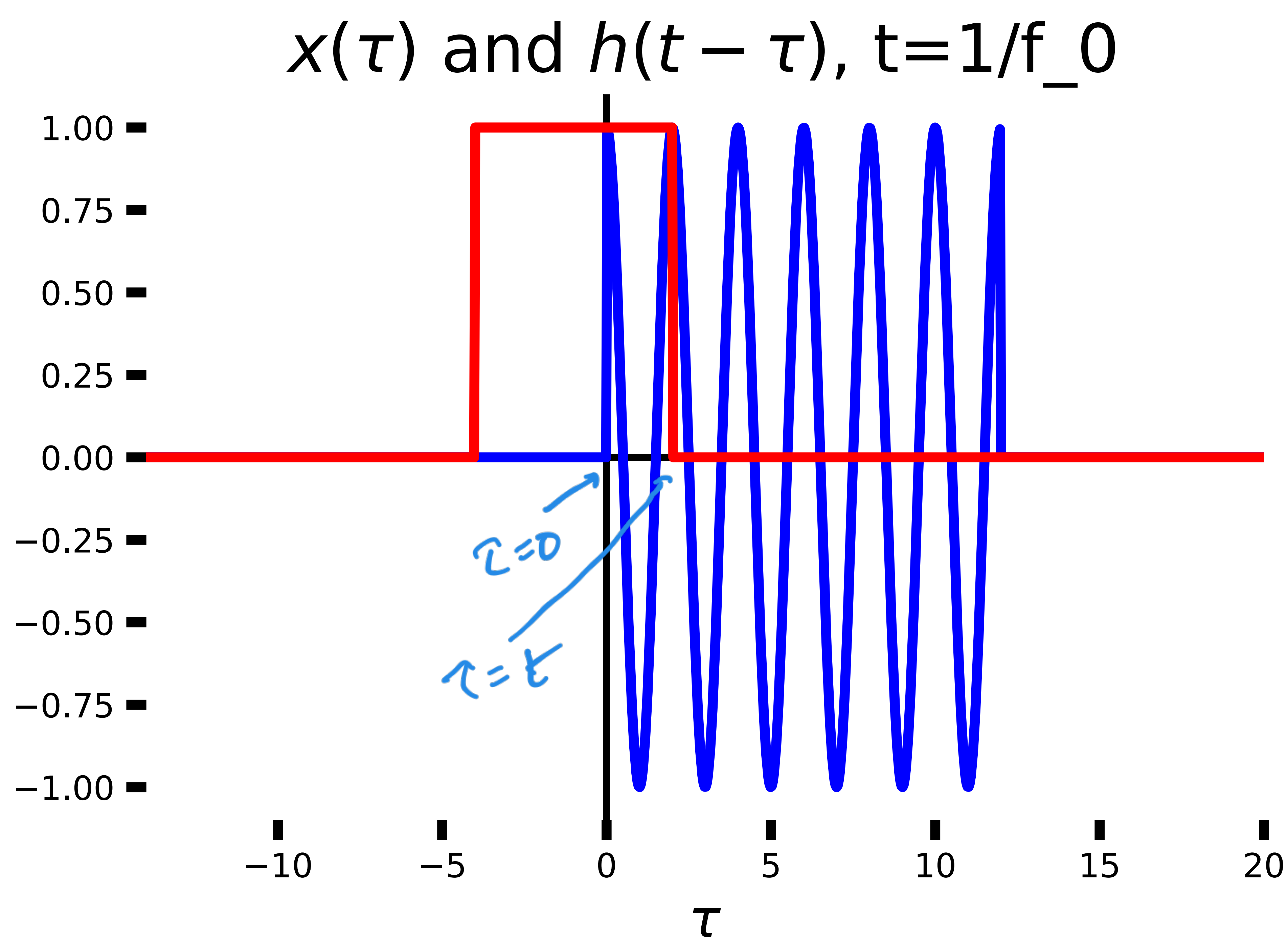


Phase 1: no overlap, $t \in (-\infty, 0]$



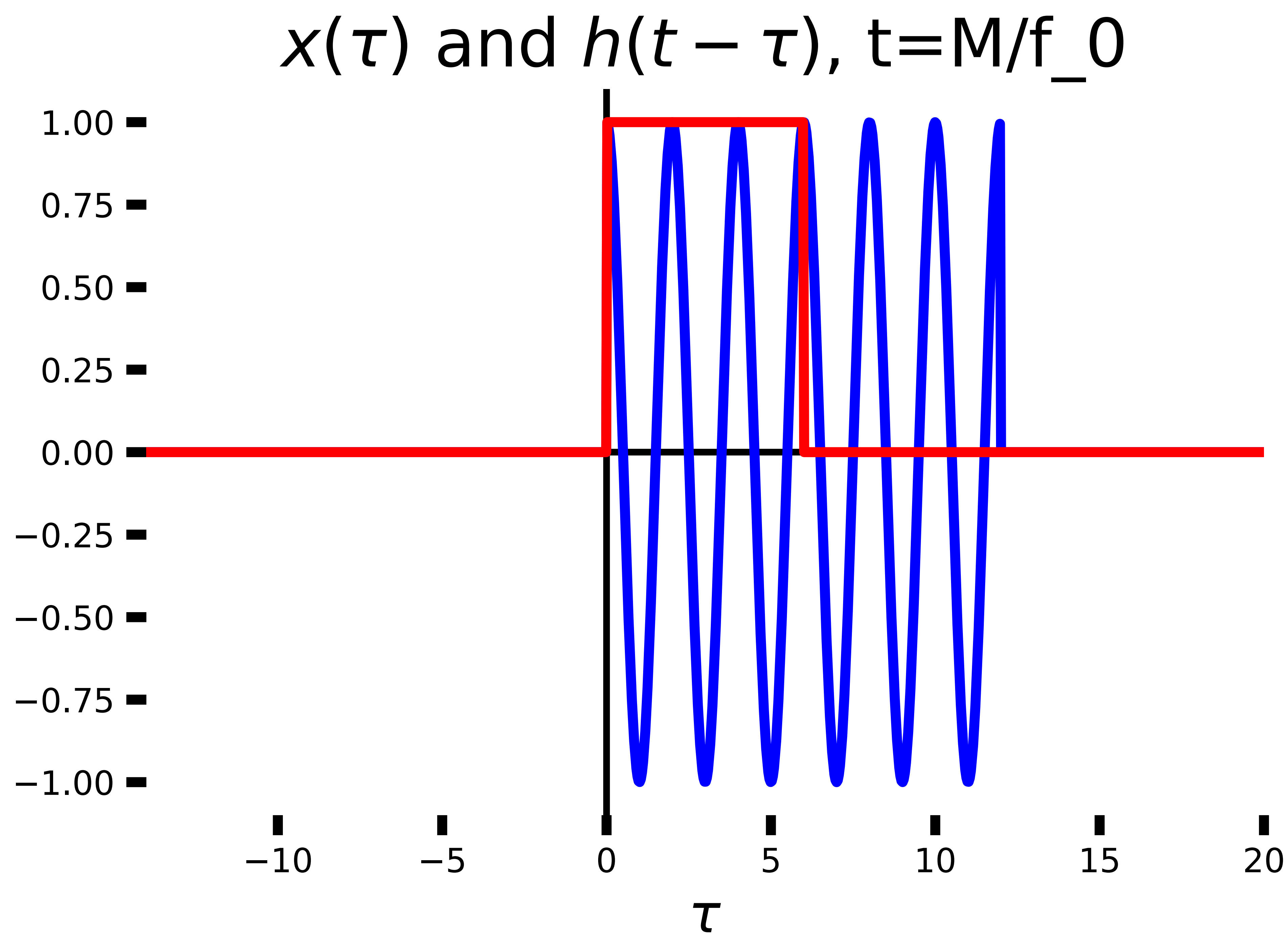
$$y(t) = 0 \tag{3}$$

Phase 2: partial overlap, $t \in (0, M/f_0]$



$$y(t) = \int_0^t \cos(2\pi f_0 \tau) d\tau = \boxed{\sin(2\pi f_0 t)} \tag{4}$$

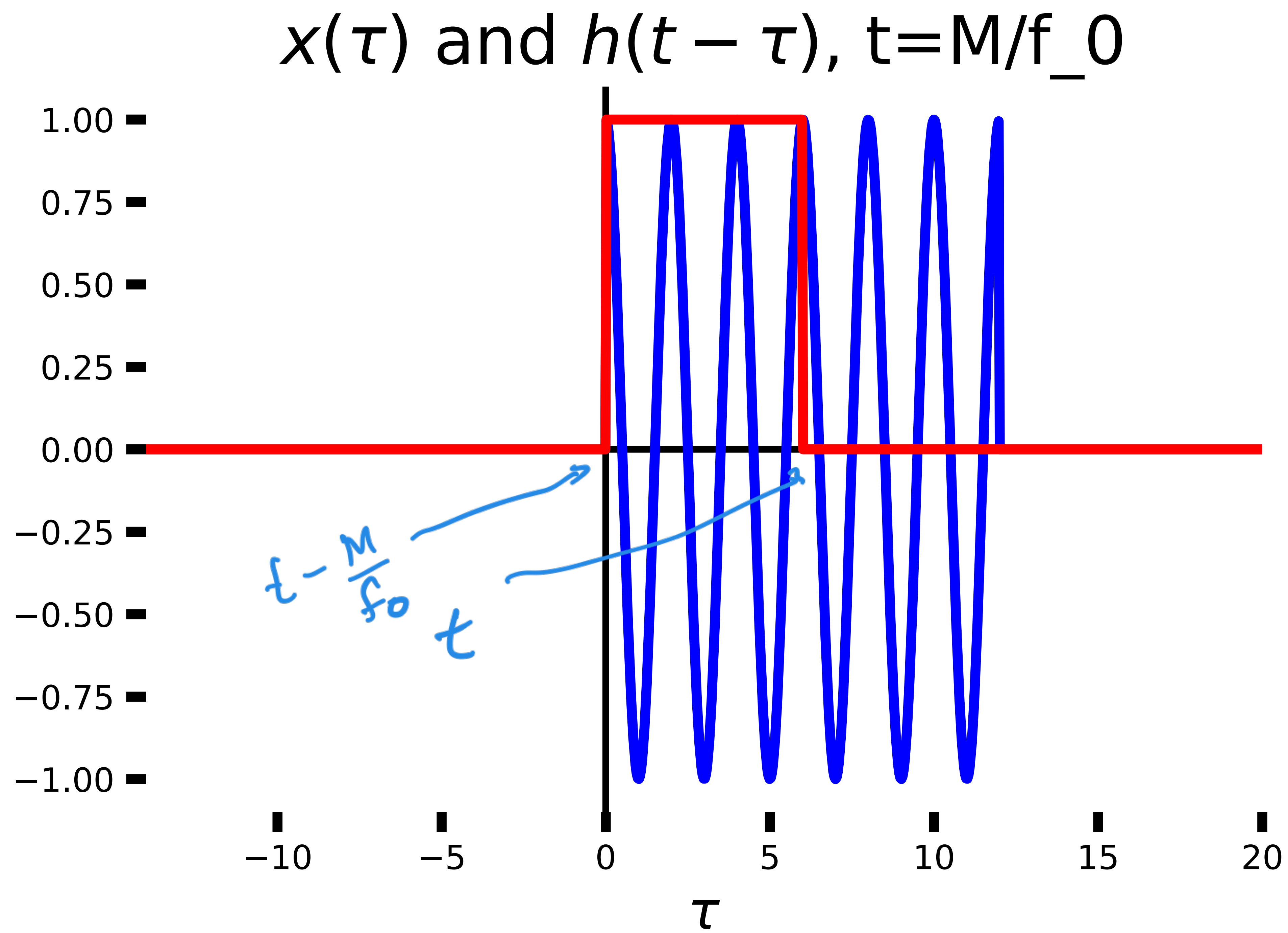
Phase 2: partial overlap, $t \in (0, M/f_0]$



$$y(t) = \int_0^t \cos(2\pi f_0 \tau) d\tau = \sin(2\pi f_0 t) \tag{4}$$

$= 0$ if $t = M/f_0$

Phase 3: total overlap, $t \in (M/f_0, K/f_0]$

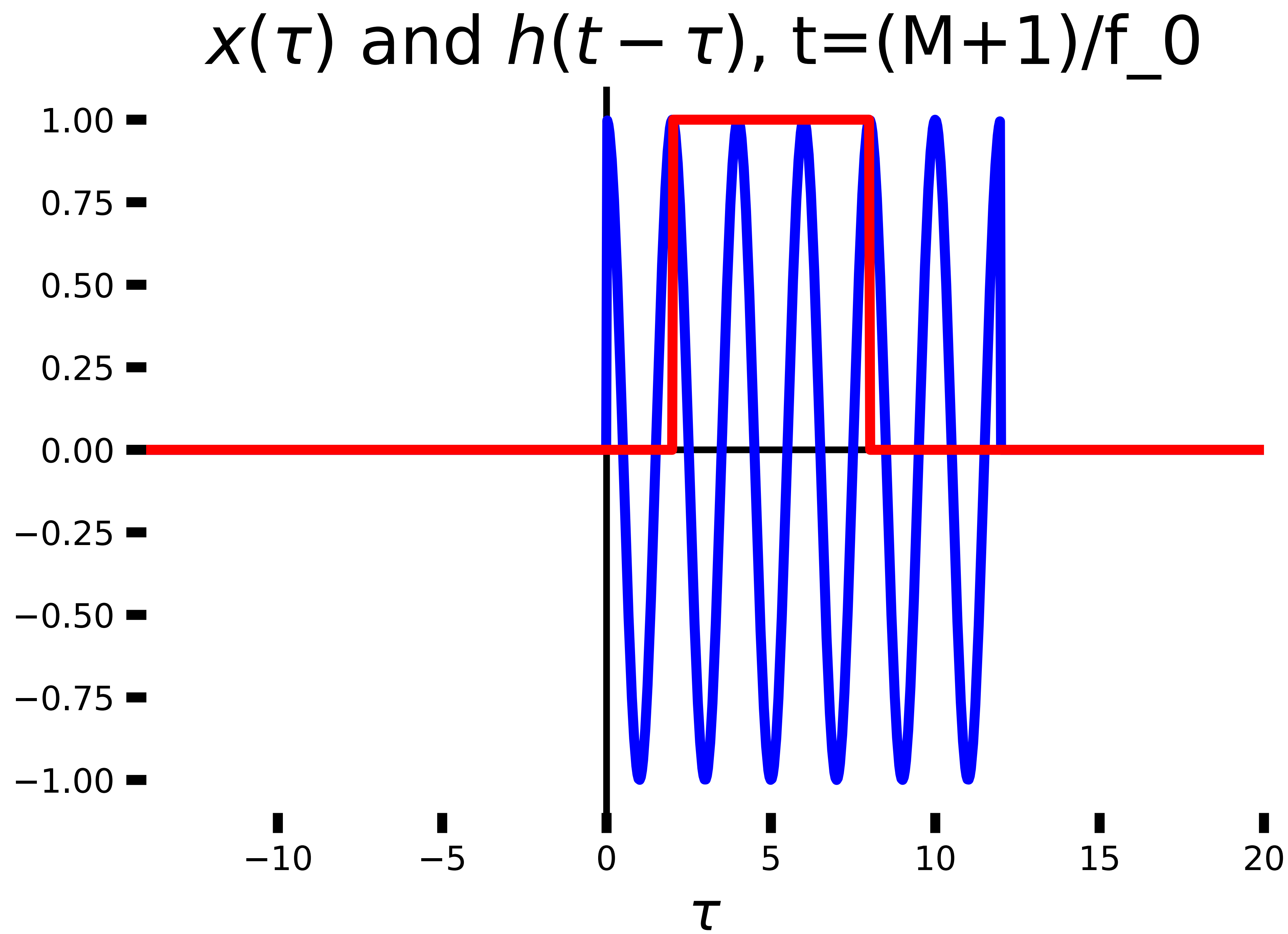


$$y(t) = \int_{t-M/f_0}^{t} \cos(2\pi f_0 \tau) d\tau = \sin(2\pi f_0 t) - \sin(2\pi f_0 (t - M/f_0))$$

$$= \sin(2\pi f_0 t) - \sin(2\pi f_0 t - 2\pi M) \boxed{= 0} \quad (5)$$

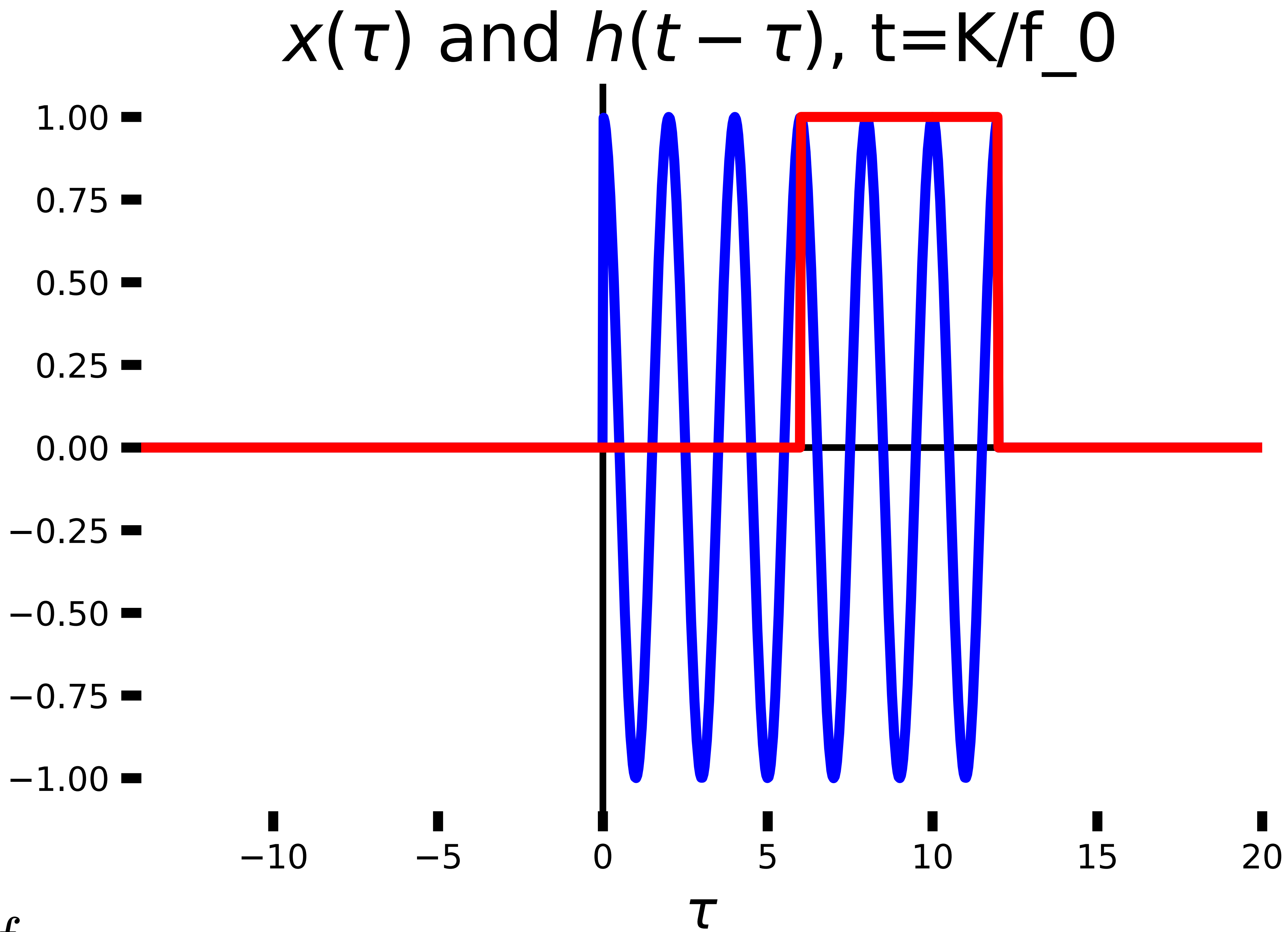
integer mult of 2π

Phase 3: total overlap, $t \in (M/f_0, K/f_0]$



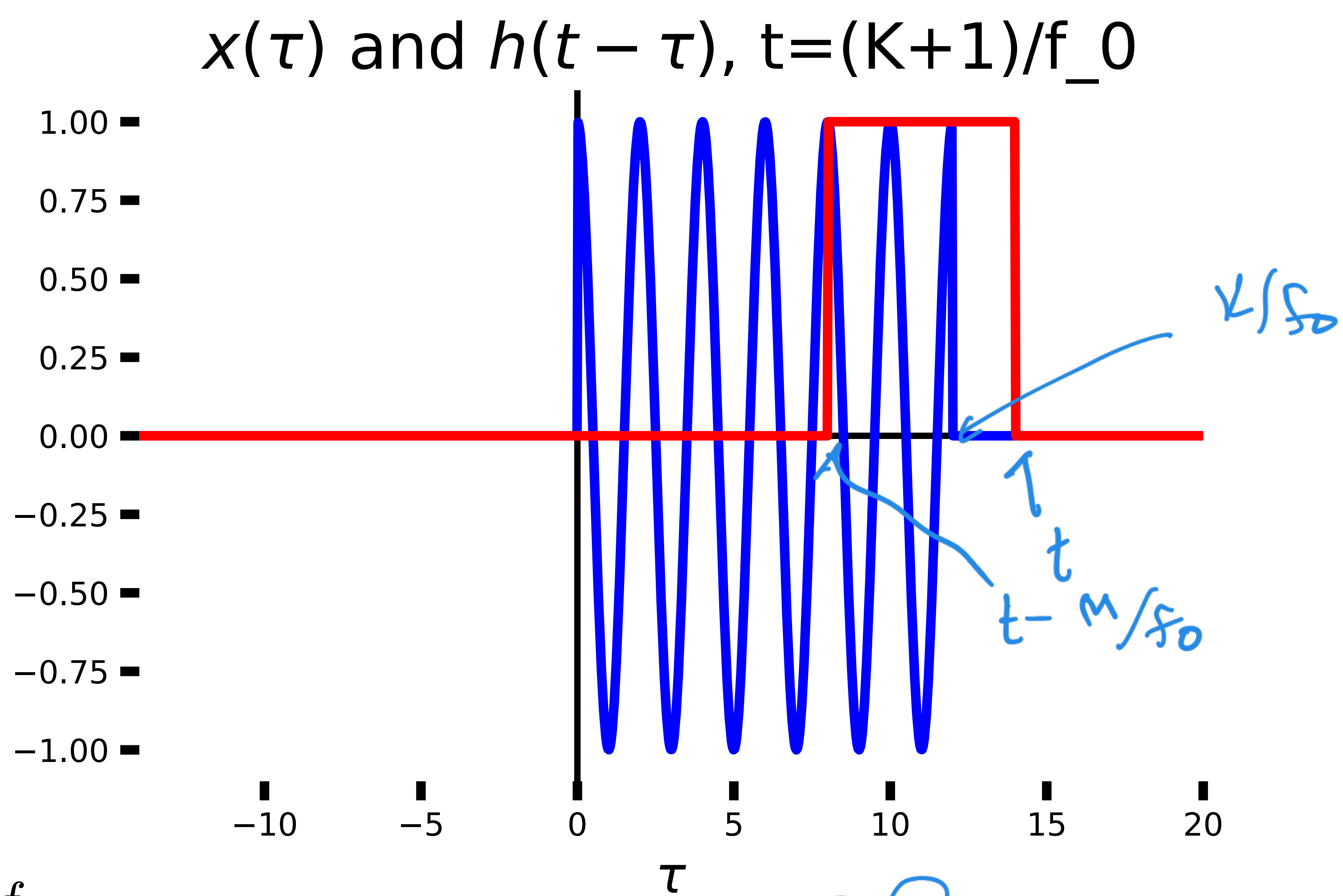
$$
\begin{aligned}
y(t) &= \int_{t-M/f_0}^{t} \cos(2\pi f_0 \tau) d\tau = \sin(2\pi f_0 t) - \sin(2\pi f_0 (t - M/f_0)) \\
&= \sin(2\pi f_0 t) - \sin(2\pi f_0 t - 2\pi M) = 0 \qquad (5)
\end{aligned}
$$

Phase 4: partial overlap, $t \in (K/f_0, (K+M)/f_0]$



$$y(t) = \int_{t-M/f_0}^{K/f_0} \cos(2\pi f_0 \tau) d\tau = \sin(2\pi K) - \sin(2\pi f_0 (t - M/f_0))$$
$$= 0 - \sin(2\pi f_0 t - 2\pi M) = -\sin(2\pi f_0 t) \tag{6}$$

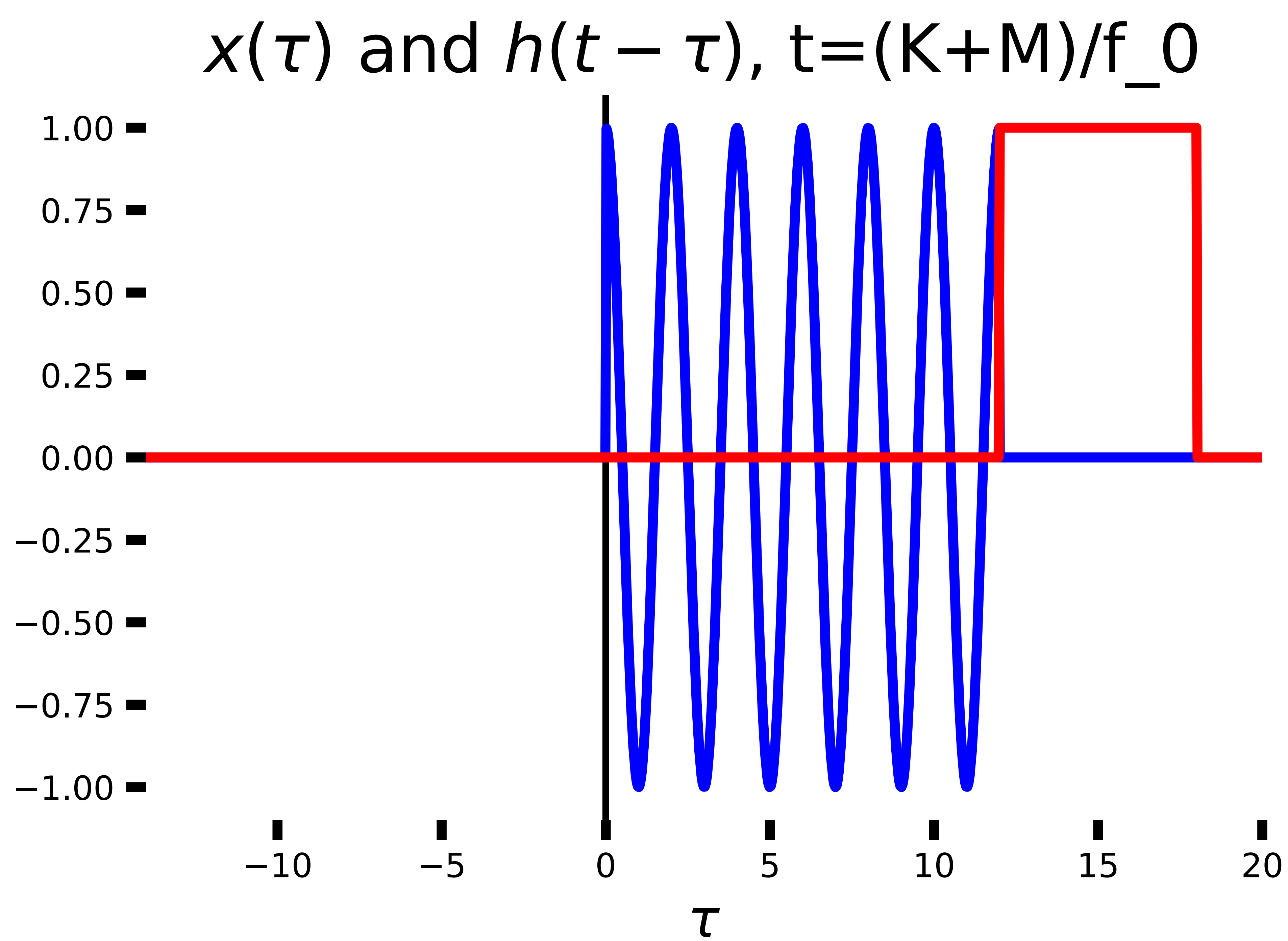
Phase 4: partial overlap, $t \in (K/f_0, (K+M)/f_0]$



$$y(t) = \int_{t-M/f_0}^{K/f_0} \cos(2\pi f_0 \tau) d\tau = \sin(2\pi K) - \sin(2\pi f_0 (t - M/f_0))$$

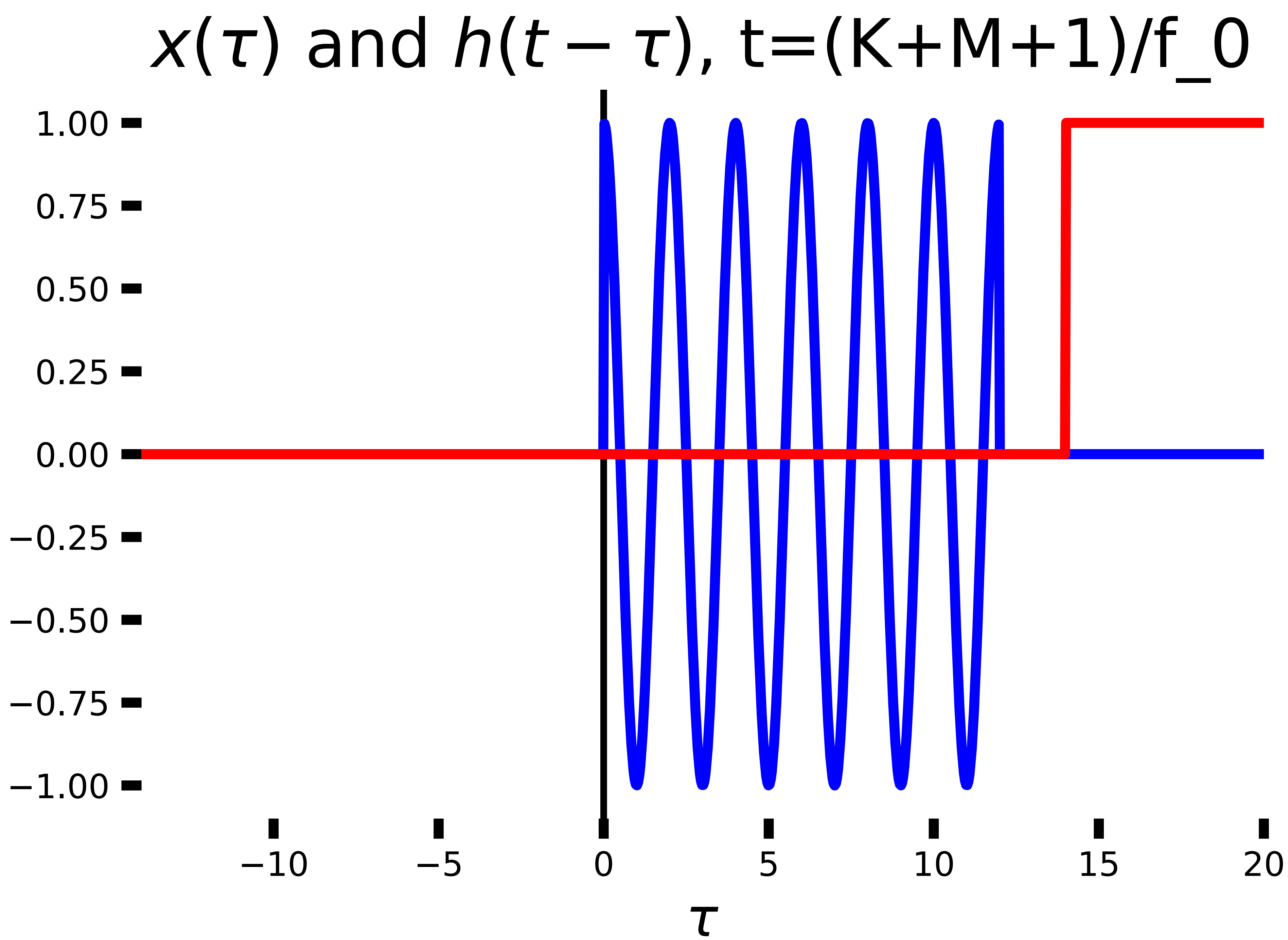
$$= 0 - \sin(2\pi f_0 t - 2\pi M) = -\sin(2\pi f_0 t) \quad (6)$$

Phase 5: no overlap, $t \in ((K+M)/f_0, \infty)$



$$y(t) = 0 \tag{7}$$

Phase 5: no overlap, $t \in ((K+M)/f_0, \infty)$

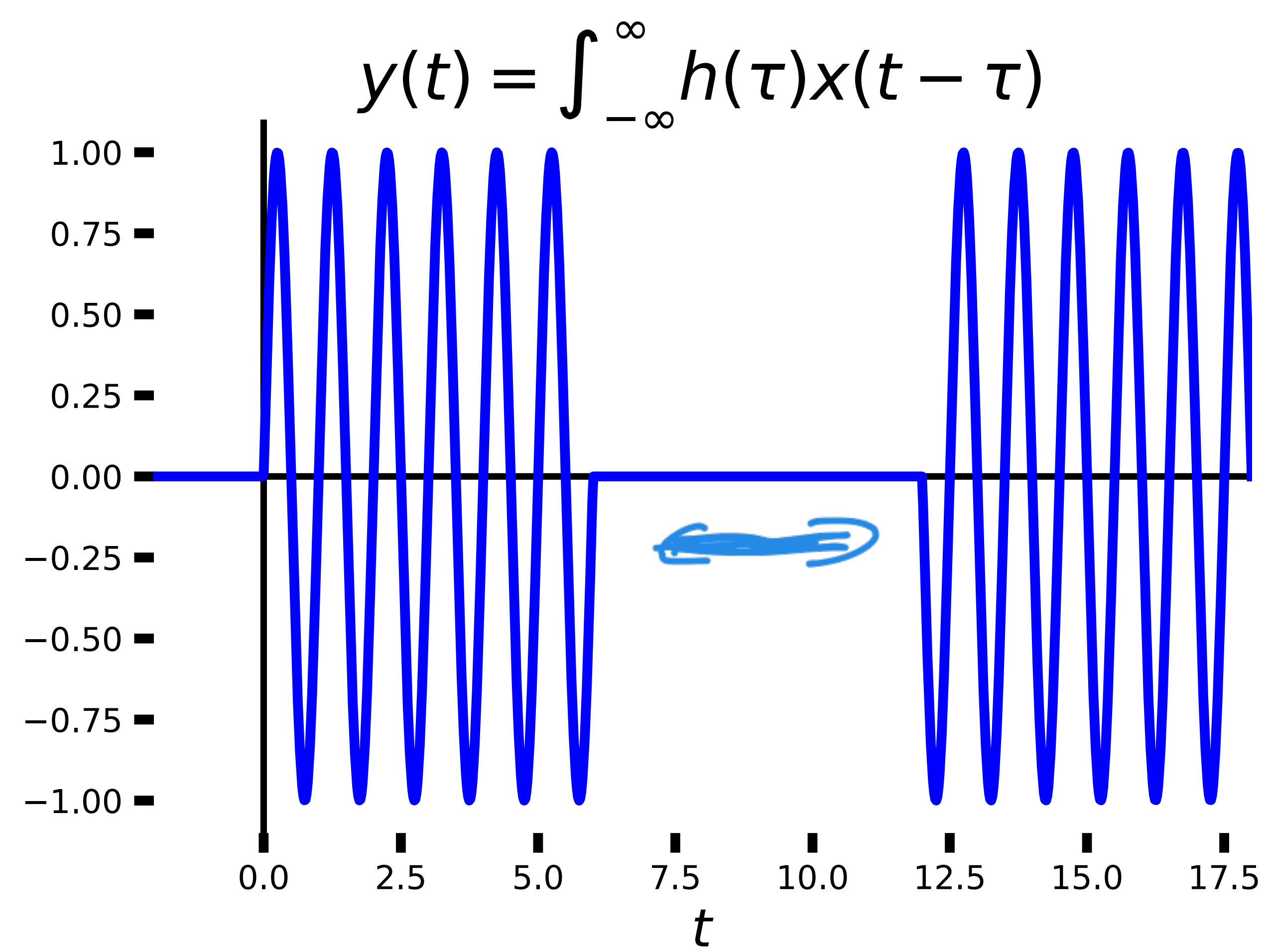


$$y(t) = 0 \tag{7}$$

Putting it all together

We therefore have

$$y(t) = \begin{cases} 0 & t \in (-\infty, 0] \\ \sin(2\pi f_0 t) & t \in (0, M/f_0] \\ 0 & t \in (M/f_0, K/f_0] \\ -\sin(2\pi f_0 t) & t \in (K/f_0, (K+M)/f_0] \\ 0 & t \in ((K+M)/f_0, \infty) \end{cases} \quad (8)$$

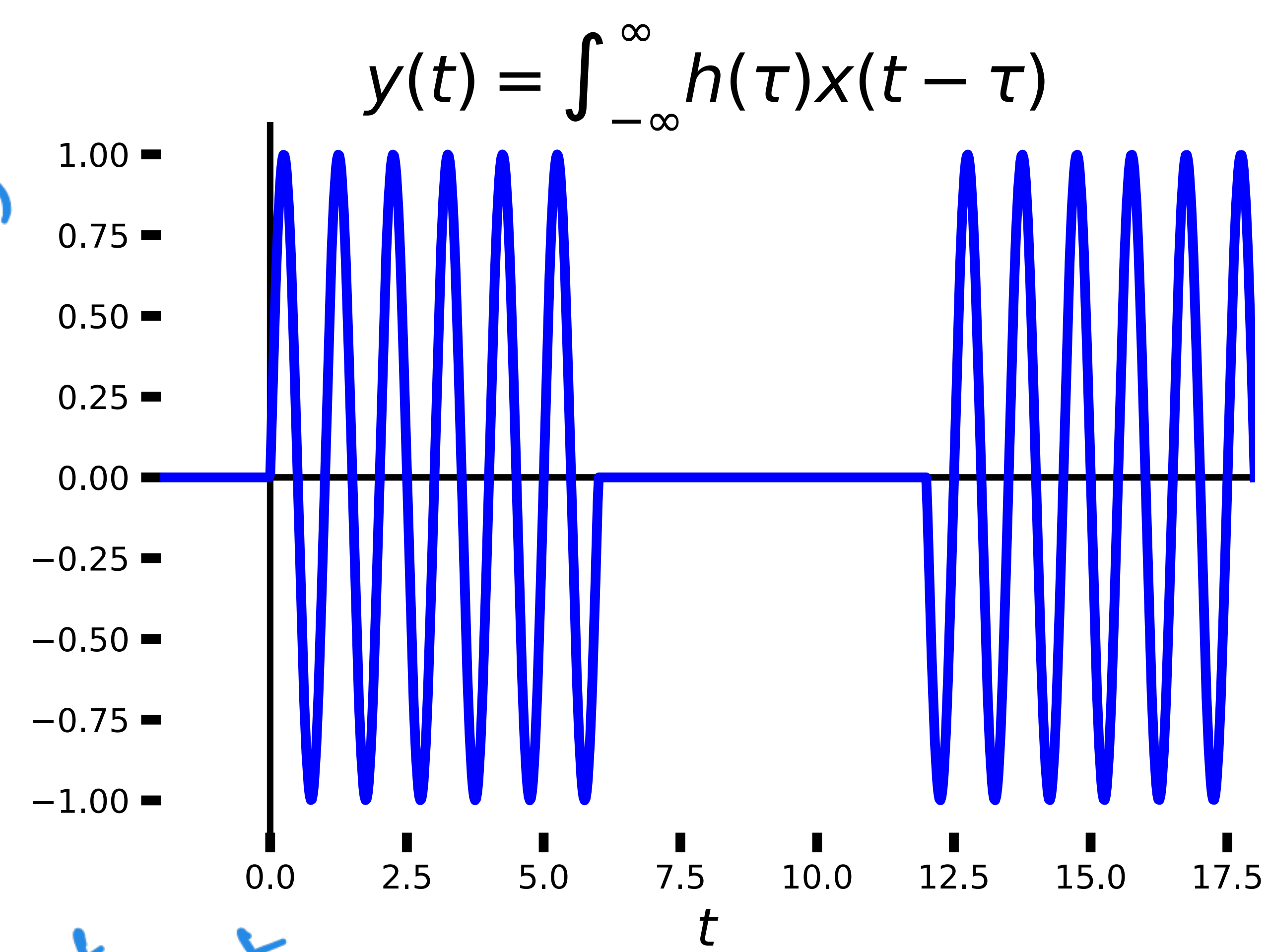


Implications

$(s(t) + x(t)) * h(t)$

legit signal

interference



$s(t) * h(t)$ might not be what we want.

The output has two sinusoidal transients and a "dead" period.

- If $K \gg M$ then the transients are short, so we will cancel out most of the interference.
- This filter will distort the legitimate (non-interference) signal.
- Is this the best way to remove a high-frequency interference? No, in general you might want to use a *notch filter* which tries to zero out a particular frequency component.

Try it yourself

Problem

Try some of these yourself to see if you get the 5 phase convolution idea:

$$x(t) = e^{-3t}(t)(u(t) - u(t-5)), h(t) = u(t) - u(t-2) \tag{9}$$

$$x(t) = \sin(300\pi t)(u(t) - u(t-1/30)), h(t) = u(t) - u(t-1/15) \tag{10}$$

$$x(t) = r(t)(u(t) - u(t-4)), h(t) = u(t-1) - u(t-6) \tag{11}$$