Linear Systems and Signals

Impulse functions in discrete time: the Kronecker $\delta[n]$

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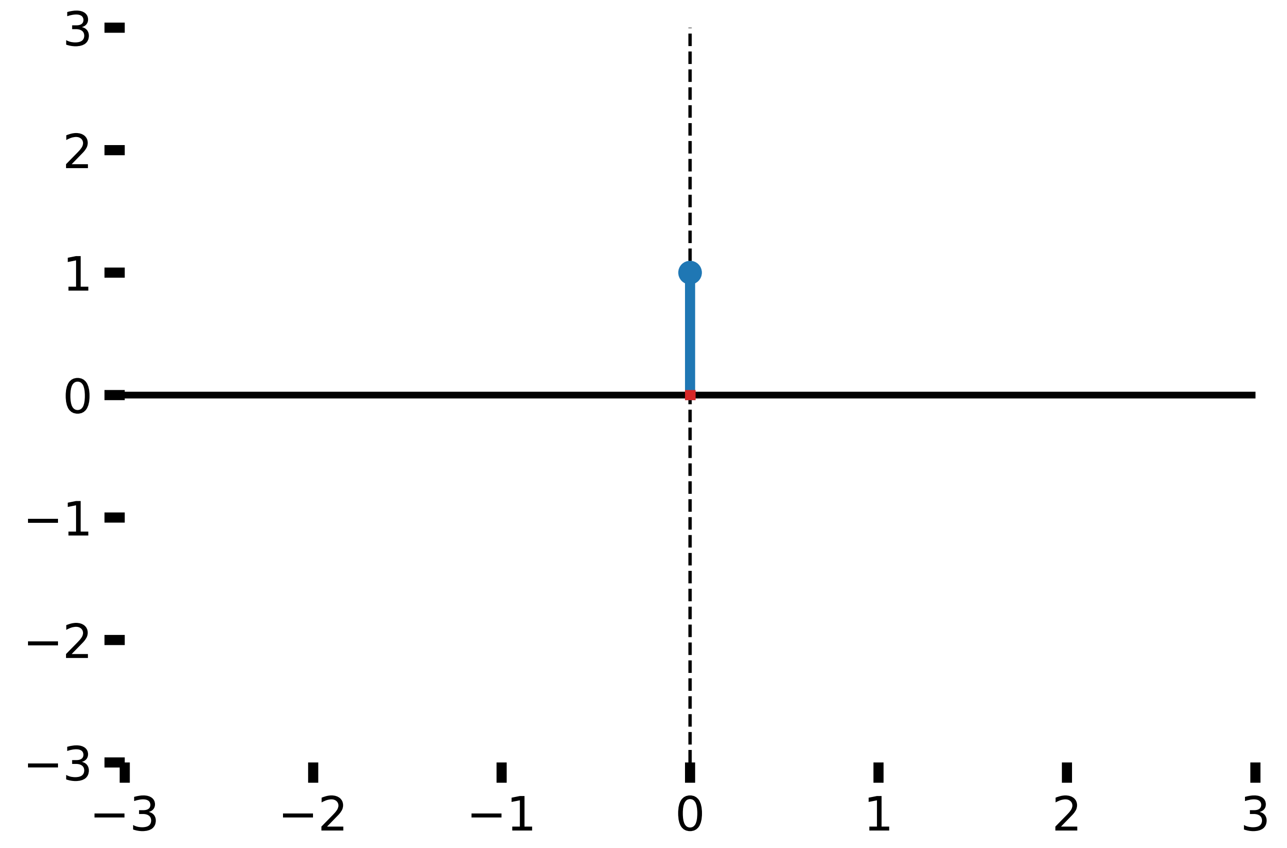
Learning objectives

The learning objectives for this section are:

- explain the Kronecker (DT) impulse function visually and graphically
- write DT signals as sums of delta functions
- use the Kronecker property to simplify multiple summations involving DT impulse functions

The Kronecker delta

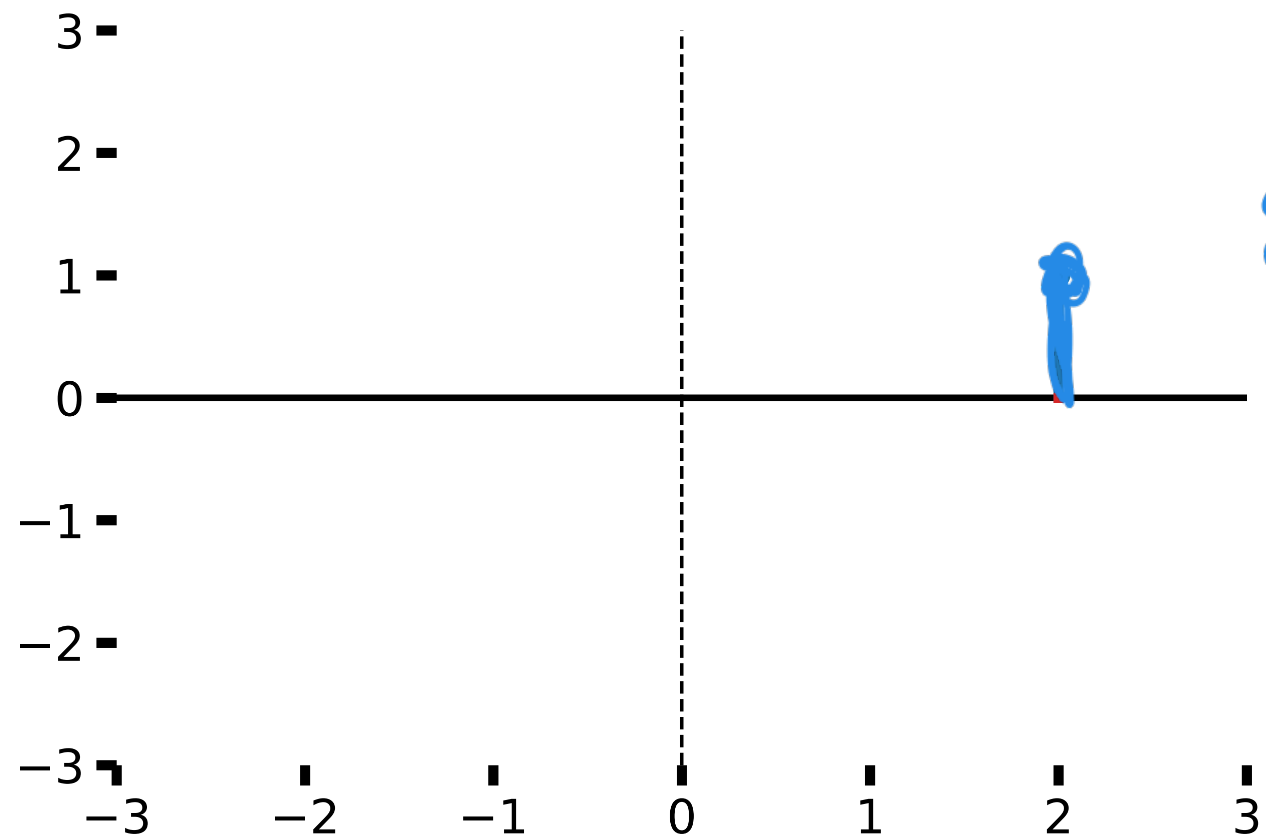


The unit impulse function, also called the Kronecker δ function, is a DT signal which is 1 at $n = 0$ and 0 elsewhere.

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \tag{1}$$

That is, $\delta[\mathrm{stuff}]$ is an *indicator* of whether $\mathrm{stuff} = 0$ or not.

Delayed delta functions



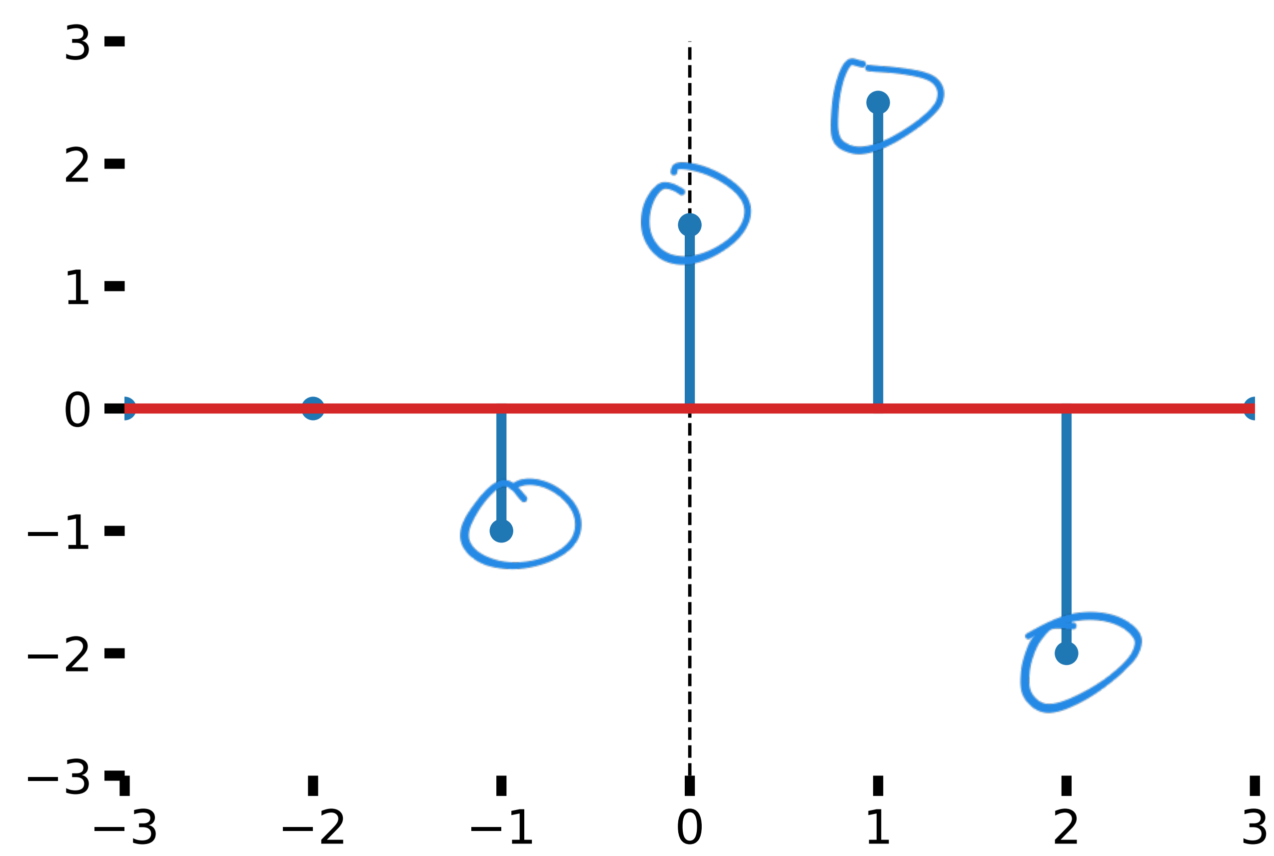
$\delta[n-2]$

when is $n-2=0$

when $n=2$!

We can move the impulse function around: $\delta[n-k]$ is a unit impulse at k.

All DT signals are made of δ-functions



$\delta[n+1] + \frac{3}{2}\delta[n]$
$+ \frac{5}{2}\delta[n-1]$
$+ 2\delta[n-2]$

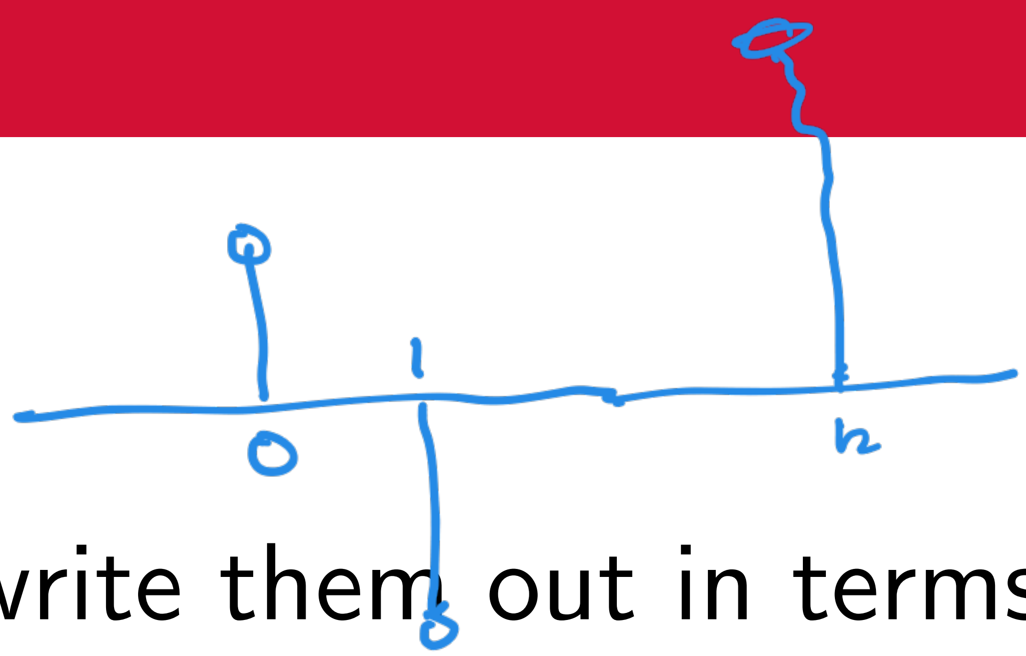
All DT signals are just sums of shifted and scaled $\delta[n]$ functions:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \qquad (2)$$

checks when $k=n$

Plugging in $n = 5$ for example, all terms in the sum are 0 except for $k = 5$. We can interpret this as the *sampling* property of $\delta[n]$.

Finite and semi-infinite sums



For finite-length DT signals we sometimes just write them out in terms of δ functions, like

$$x[n] = 2\delta[n] - 3\delta[n-1] + 4\delta[n-k] \tag{3}$$

The DT unit step function is

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{4}$$

Dealing with multiple indices

A common situation is to have double or triple sums involving many δ functions. That is where things can get a bit tricky. You have to simplify things by figuring out which terms "survive":

$$h[n] = \sum_{k=-\infty}^{\infty} h[k]\delta[n-k] \tag{5}$$

$$x[n] = \sum_{\ell=-\infty}^{\infty} x[\ell]\delta[n-\ell] \tag{6}$$

DT Convolution

$$(h * x)[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \tag{7}$$

$$= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \tag{8}$$

Example: $h[n] = \left(\frac{1}{2}\right)^n u[n]$ and $x[n] = u[n]$.

Multiple indices

Writing it out:

$$y[n] = \sum_{m=-\infty}^{\infty} \overbrace{\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \delta[m-k]}^{h} \overbrace{\sum_{\ell=0}^{\infty} \delta[n-m-\ell]}^{x} \tag{9}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \left(\frac{1}{2}\right)^k \delta[m-k]\delta[n-m-\ell] \tag{10}$$

First: $\delta[m-k] \implies m-k=0 \implies m=k$:

$$y[n] = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \left(\frac{1}{2}\right)^k \delta[n-k-\ell] \tag{11}$$

Multiple indices

Starting from:

$$y[n] = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \left(\frac{1}{2}\right)^k \delta[n-k-\ell] \tag{12}$$

$\Rightarrow$ when $n-k-\ell=0$

Next: $\delta[n-k-\ell] \implies \ell = n-k$, if $k = 0, 1, 2, \ldots$ we have $\ell = n, n-1, n-2, \ldots$. So the sum on k can only go $k = 0$ to n:

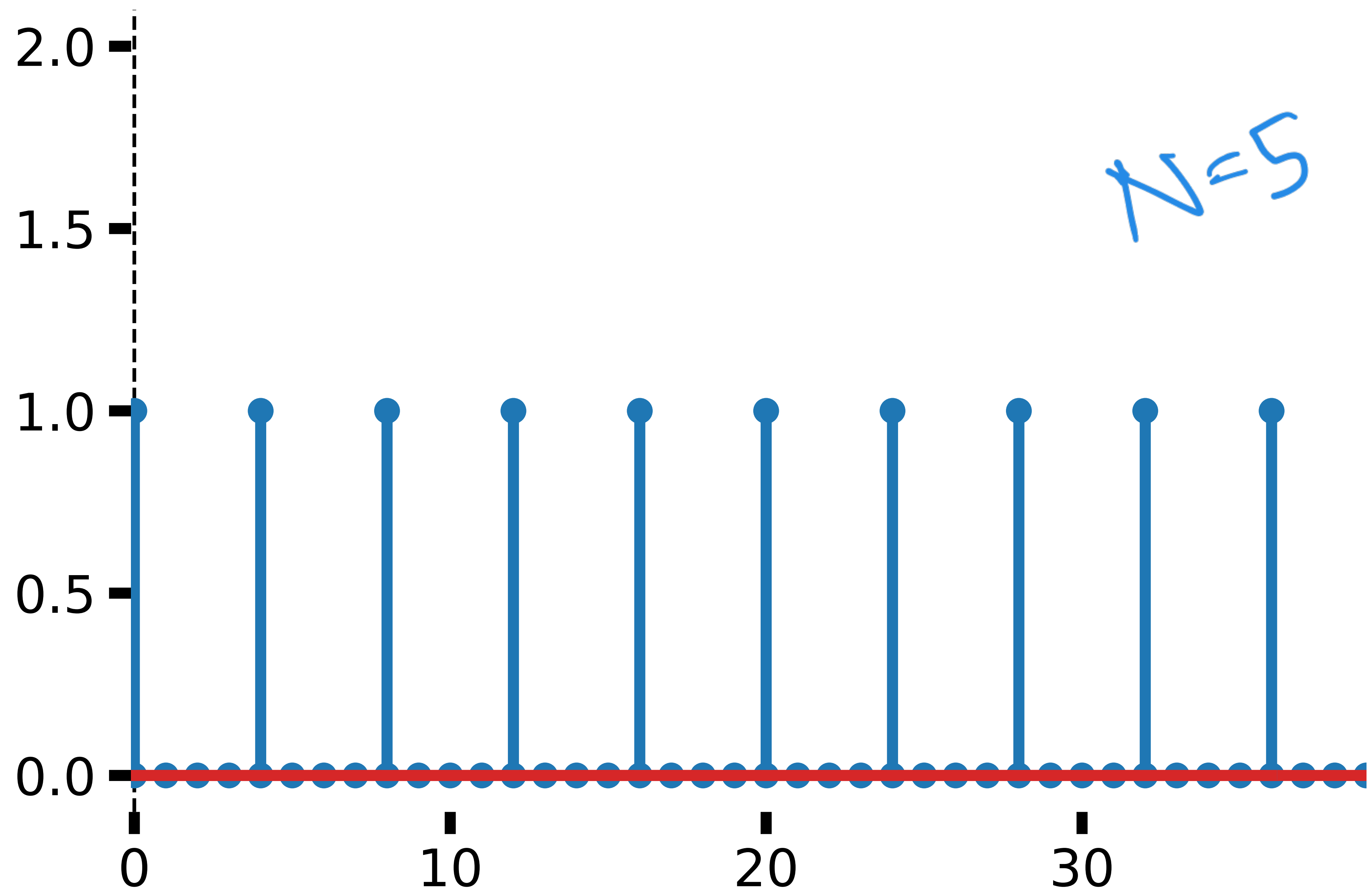
$$y[n] = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \tag{13}$$

$$= \frac{1 - (\frac{1}{2})^{n+1}}{1 - \frac{1}{2}} \tag{14}$$

partial geometric series

$=$ algebra ...

Delaying and making copies



We can also make *impulse trains* in DT:

$= 0$ when $n = kN$
n is an integer mult. of N

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]. \tag{15}$$

Try it yourself

Problem

Try writing down the following signals in terms of $\delta[n]$:

- *A signal which is 5 at time -2, -3 at time 0, and 1 at time 1.*
- *A square wave with period 6.*
- *A triangle wave with period 8.*