Linear Systems and Signals

Impulse functions in continuous time: the Dirac $\delta(t)$

Anand D. Sarwate

Department of Electrical and Computer Engineering
Rutgers, The State University of New Jersey

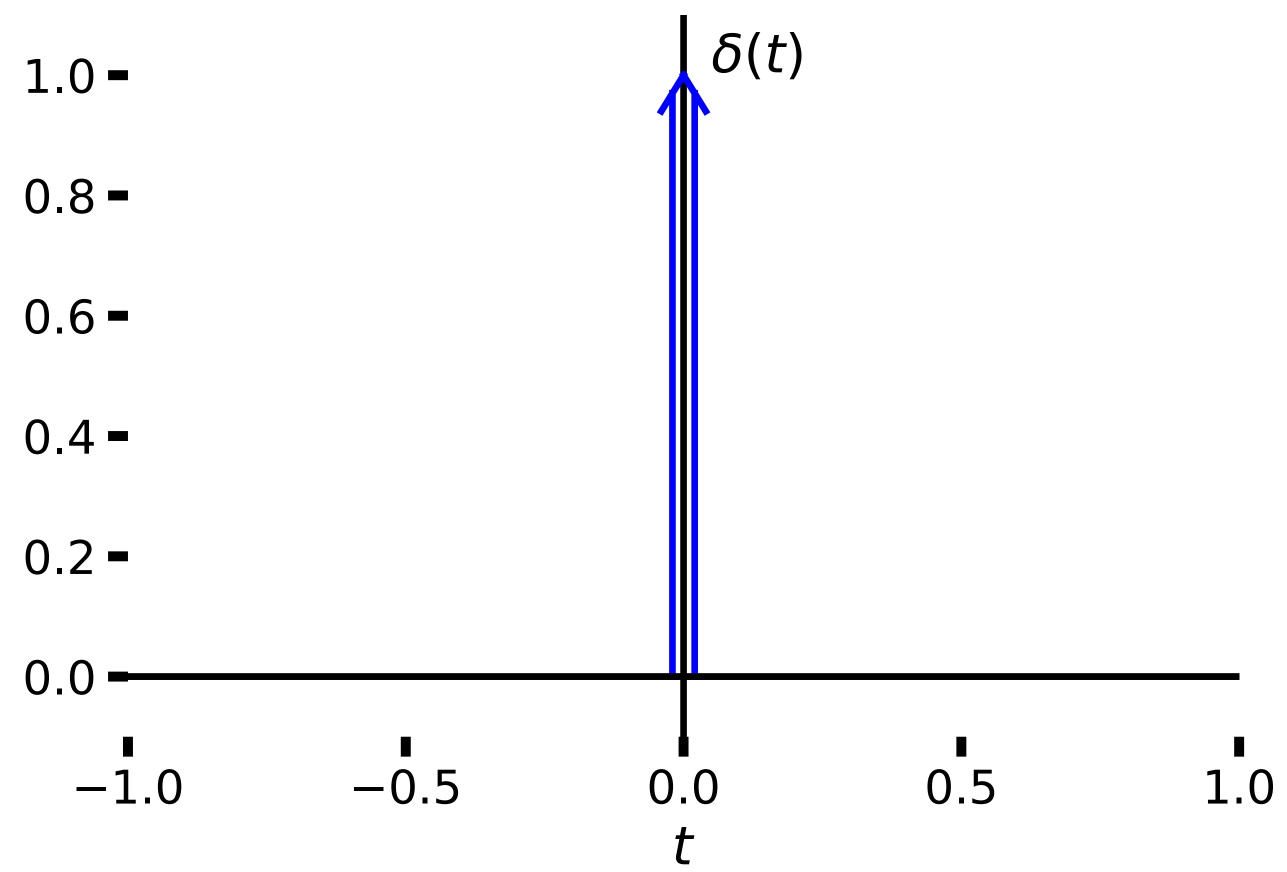
2020

Learning objectives

The learning objectives for this section are:

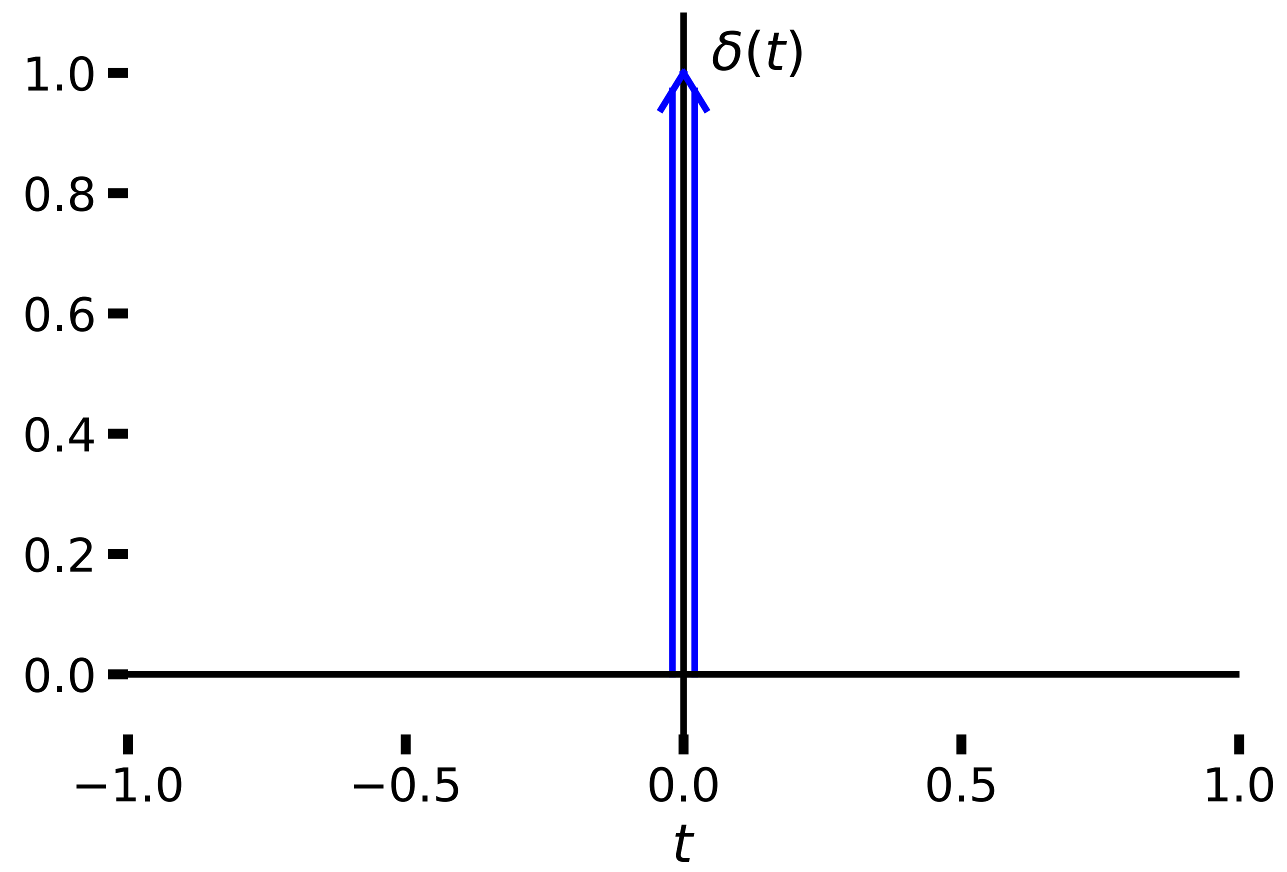
- explain the unit-area in zero-time property of Dirac (CT) impulse function as a limit of box functions
- apply the sifting property of the Dirac delta function
- use impulse trains to periodically sample a signal

Unit impulses as generalized functions



There are many ways to interpret the unit impulse function in continuous time. It is what is called a *generalized* function (or distribution) and getting a rigorous mathematical treatment is a little beyond the scope of this class.

How to think about the δ-function

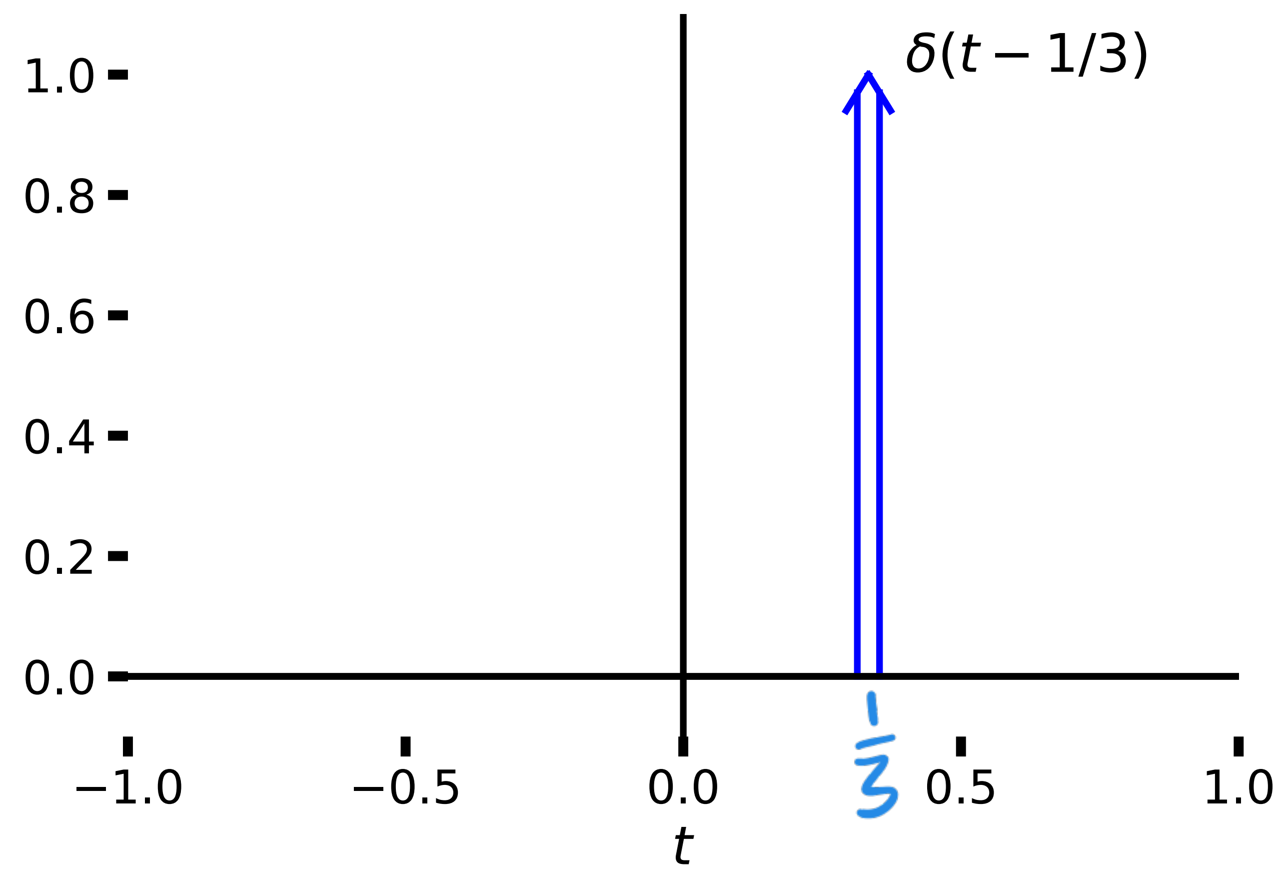


Think of $\delta(t)$ as a function that has an "area under the curve" of 1 entirely concentrated at $t = 0$. So it only "goes into action" when it appears in an integral.

Two ideas should come to mind when you see $\delta(t)$: something is being *sampled* or something is being *"copied"*.

Shifting delta functions

when is $t - 1/3 = 0$?

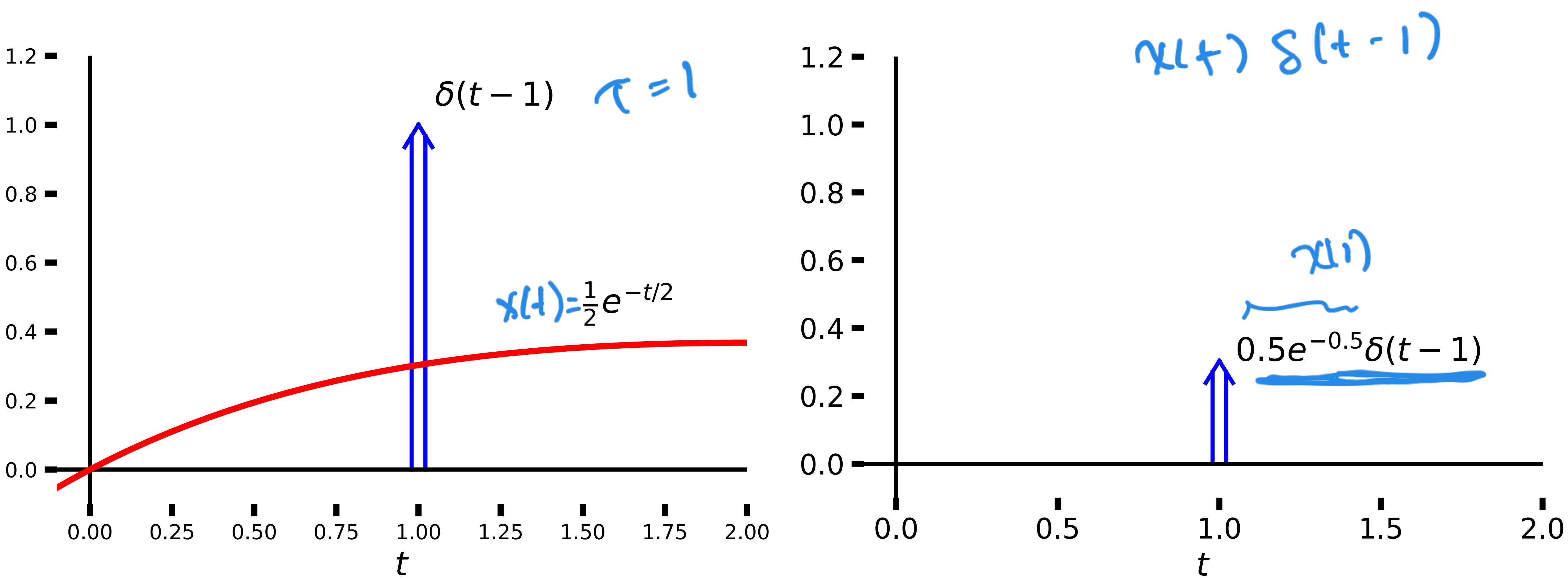


Time and amplitude shifts work the same way:

$$\alpha\delta(t-\tau) \tag{1}$$

acts as a total area α concentrated at $t = \tau$

Sampling property

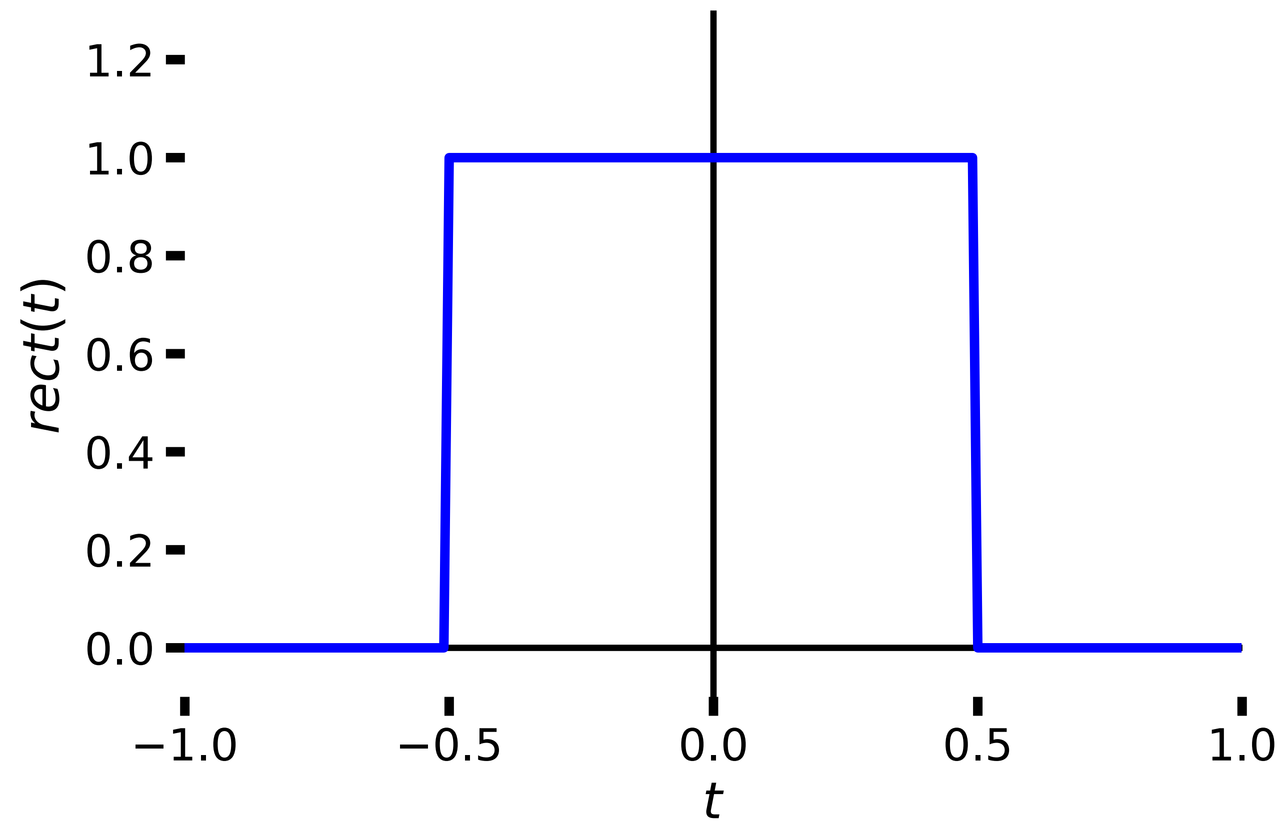


This is the *sifting* or *sampling* property of the $\delta(t)$:

$$x(\tau) = \int_{-\infty}^{\infty} x(t)\delta(t-\tau)dt \qquad (2)$$

The unit area is scaled by the function value at τ – the product $x(t)\delta(t-\tau)$ has area $x(t)$ at $t = \tau$. When we integrate we get this area $x(\tau)$ and that's it.

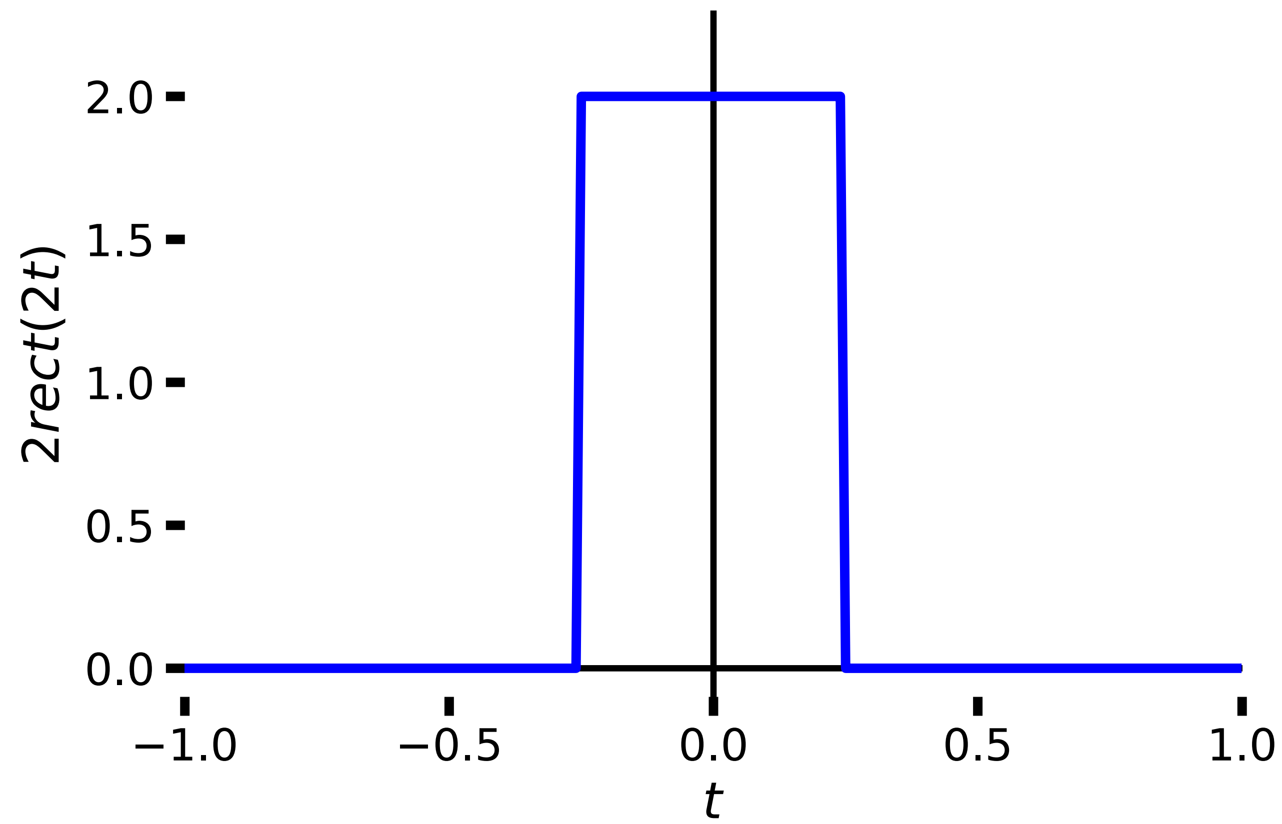
Intuition as a limit



Intuitively we can think of $\delta(t)$ as a limit of rectangles

$$\delta(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \operatorname{rect}\left(\frac{t}{\epsilon}\right) \tag{3}$$

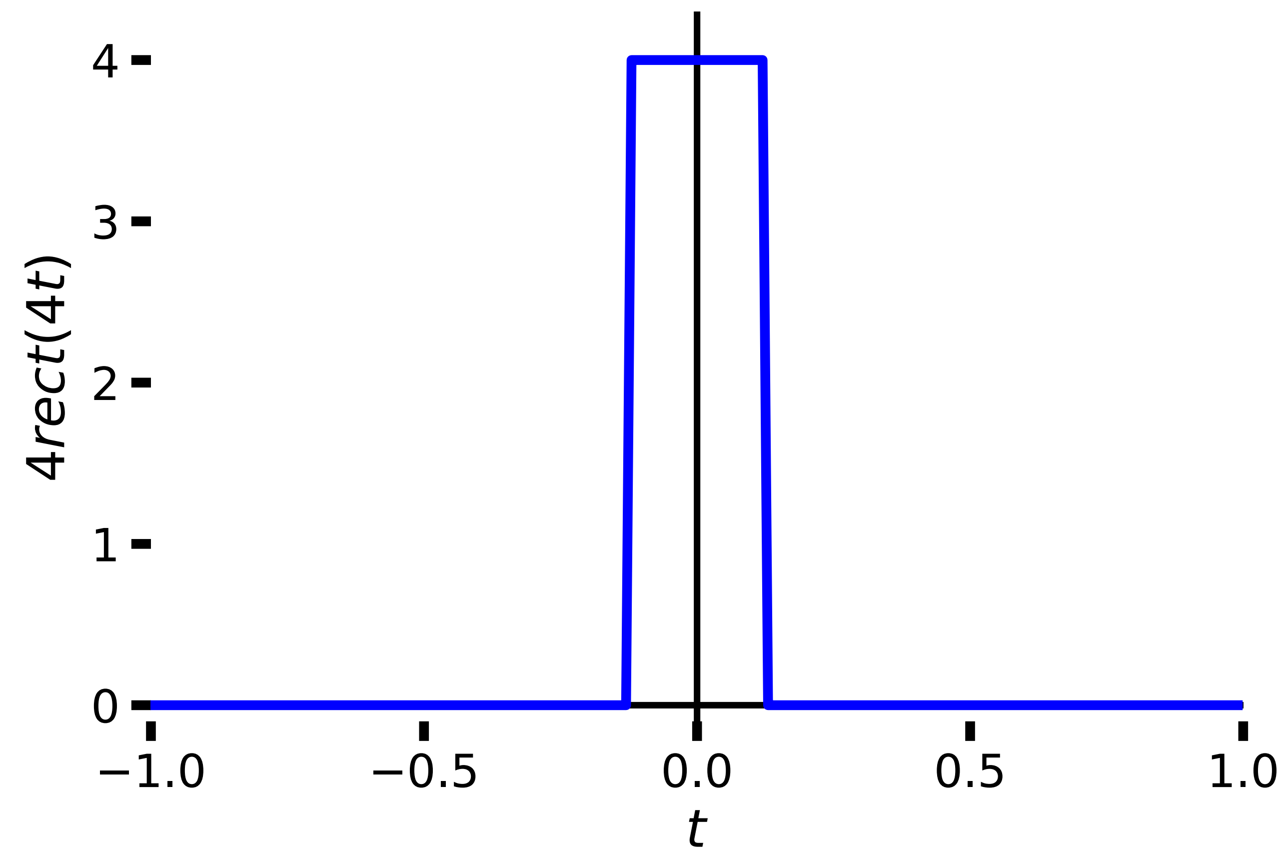
Intuition as a limit



Intuitively we can think of $\delta(t)$ as a limit of rectangles

$$\delta(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \operatorname{rect}\left(\frac{t}{\epsilon}\right) \tag{3}$$

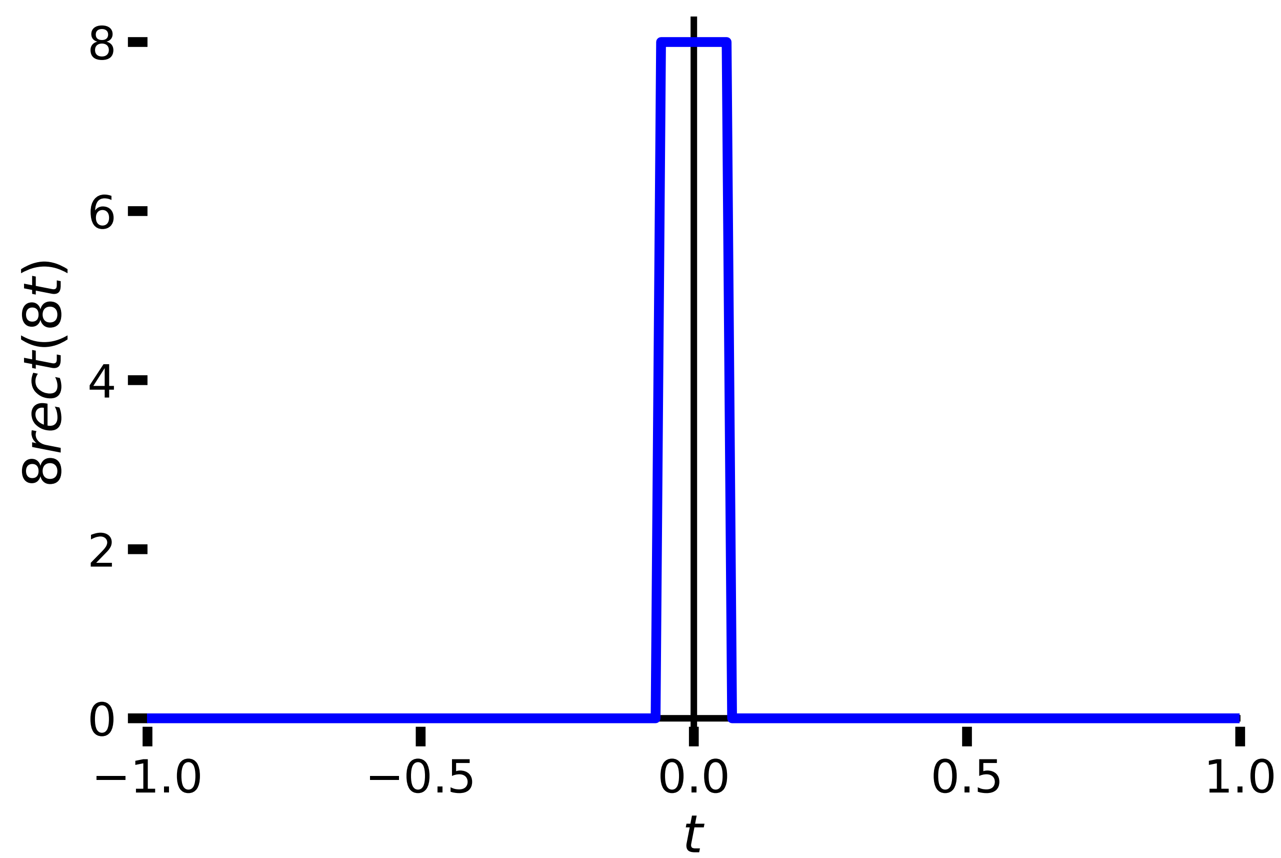
Intuition as a limit



Intuitively we can think of $\delta(t)$ as a limit of rectangles

$$\delta(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \operatorname{rect}\left(\frac{t}{\epsilon}\right) \tag{3}$$

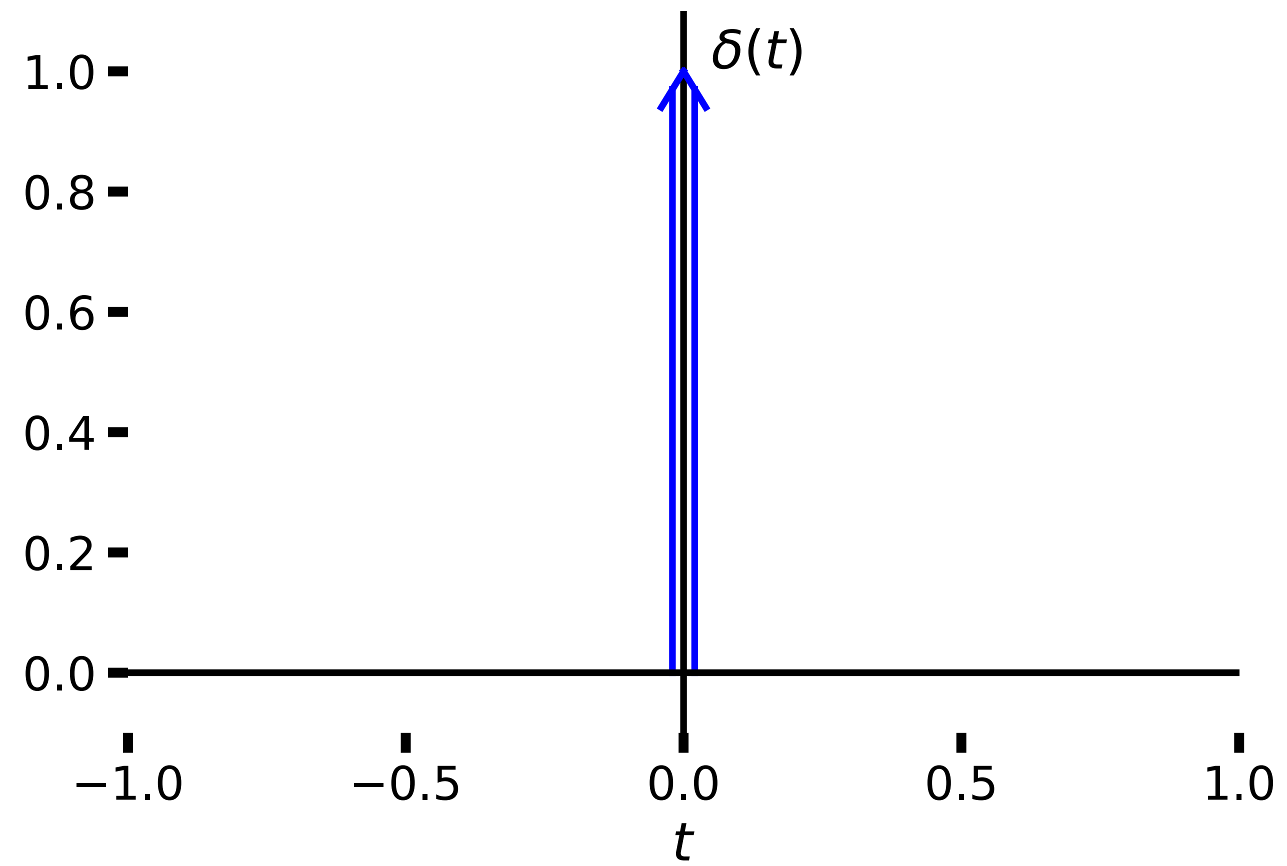
Intuition as a limit



Intuitively we can think of $\delta(t)$ as a limit of rectangles

$$\delta(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \operatorname{rect}\left(\frac{t}{\epsilon}\right) \tag{3}$$

Intuition as a limit



Intuitively we can think of $\delta(t)$ as a limit of rectangles

$$\delta(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \operatorname{rect}\left(\frac{t}{\epsilon}\right) \tag{3}$$

Time scaling

How does time-scaling (dilation/compression) affect $\delta(t)$? Looking a the previous limit of rectangles can help:

1. If we replace $\mathrm{rect}(t) \to \mathrm{rect}(at)$ then the rectangle is of height 1 but width $\frac{1}{|a|}$, so the total area is $\frac{1}{|a|}$.
2. Taking the limit, we get

$$\delta(at) = \frac{1}{|a|}\delta(t) \tag{4}$$

Extending this:

$$\delta(at - b) = \delta\left(a\left(t - \frac{b}{a}\right)\right) = \frac{1}{|a|}\delta\left(t - \frac{b}{a}\right) \tag{5}$$

An example

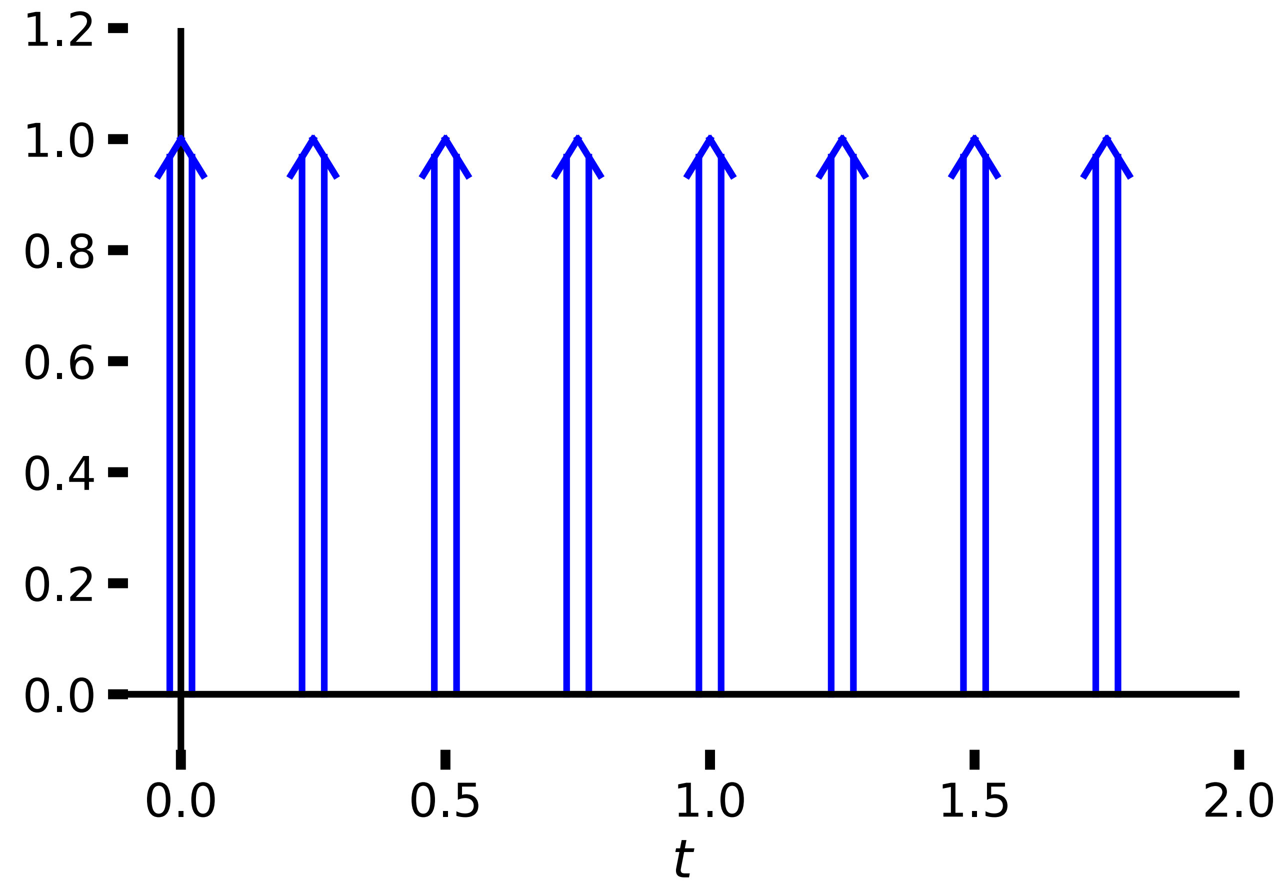
Problem

Use the sampling property of the impulse function to evaluate the integral

$$\int_{t=\cancel{-\infty}\;0}^{\infty} \overbrace{\frac{1}{2}e^{-t/2}}^{\text{what is this at } t=8} \underline{u(t)}\,\delta\left(\frac{t}{4}-2\right) dt \tag{6}$$

1. Rewrite the δ function: $\delta\left(\frac{t}{4}-2\right) = 4\delta(t-8)$.
2. Apply the $u(t)$ window to the integral.
3. Sample the function at the location of the δ, which is $t = 8$.

Impulse trains

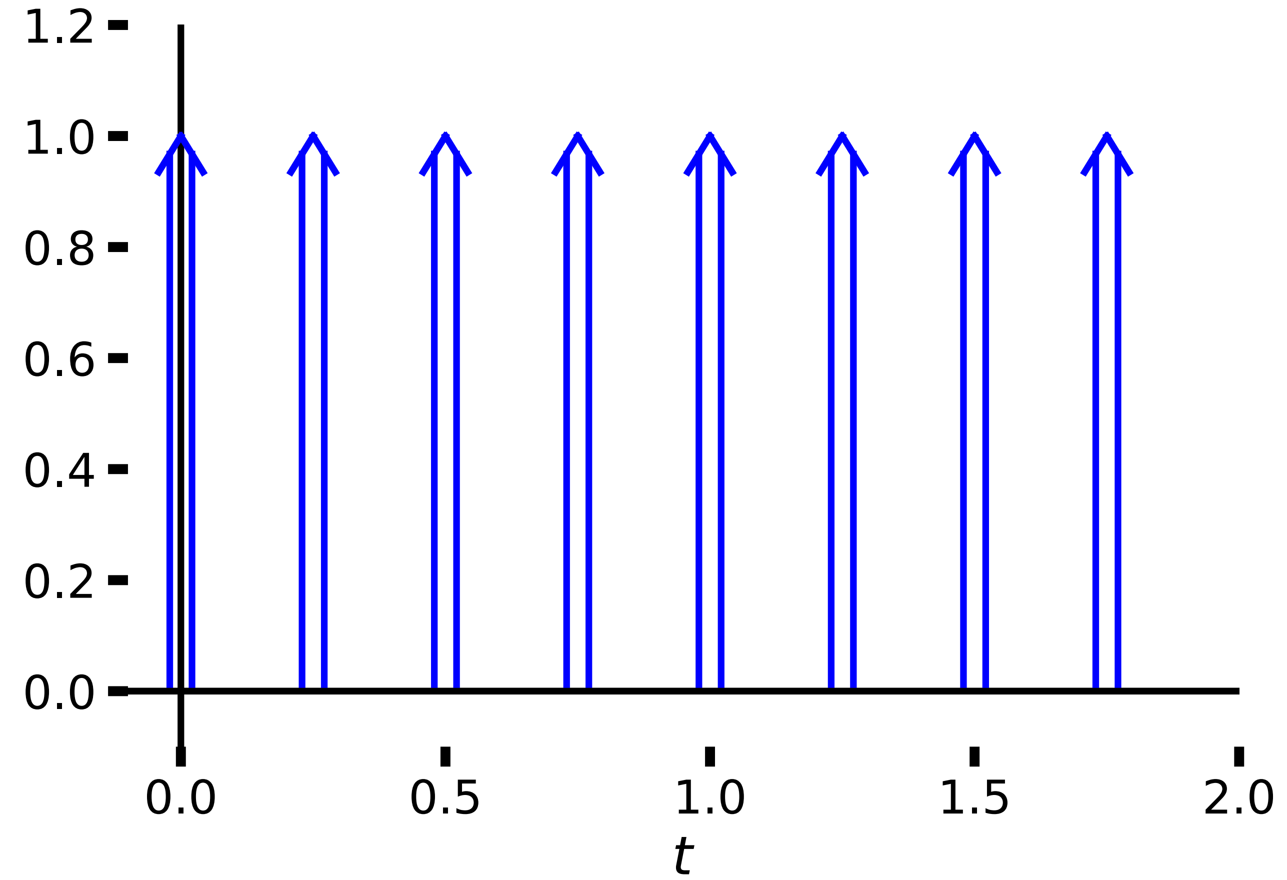


$T_0 = \frac{1}{4}$

An *impulse train* is two-sided signal containing evenly-spaced δ functions:

$$g(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0). \qquad (7)$$

Impulse trains and sampling

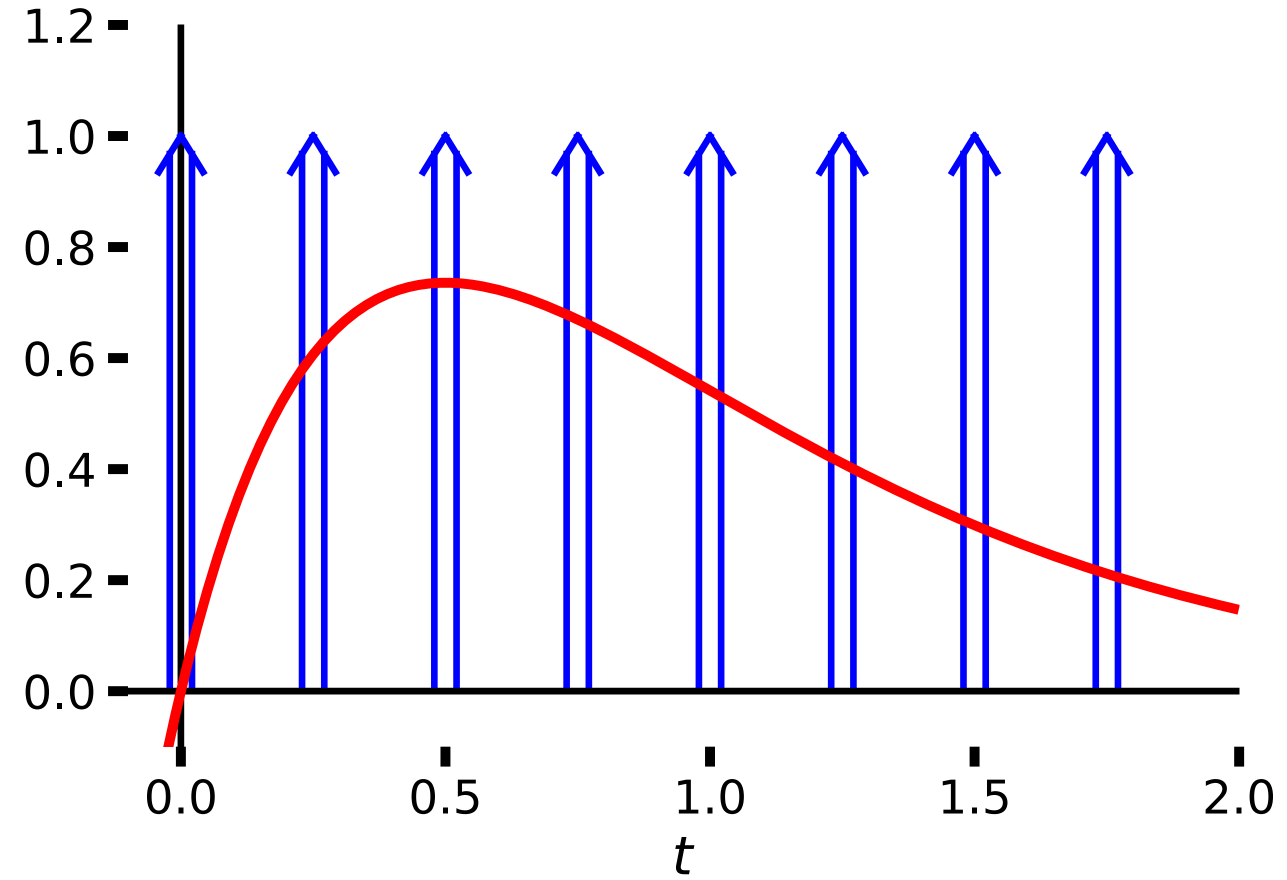


Impulse trains can be used to model a sampled signal:

$$x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT_0) = \sum_{k=-\infty}^{\infty} x(kT_0)\delta(t - kT_0) \quad (8)$$

δ[n]
(Kronecker)

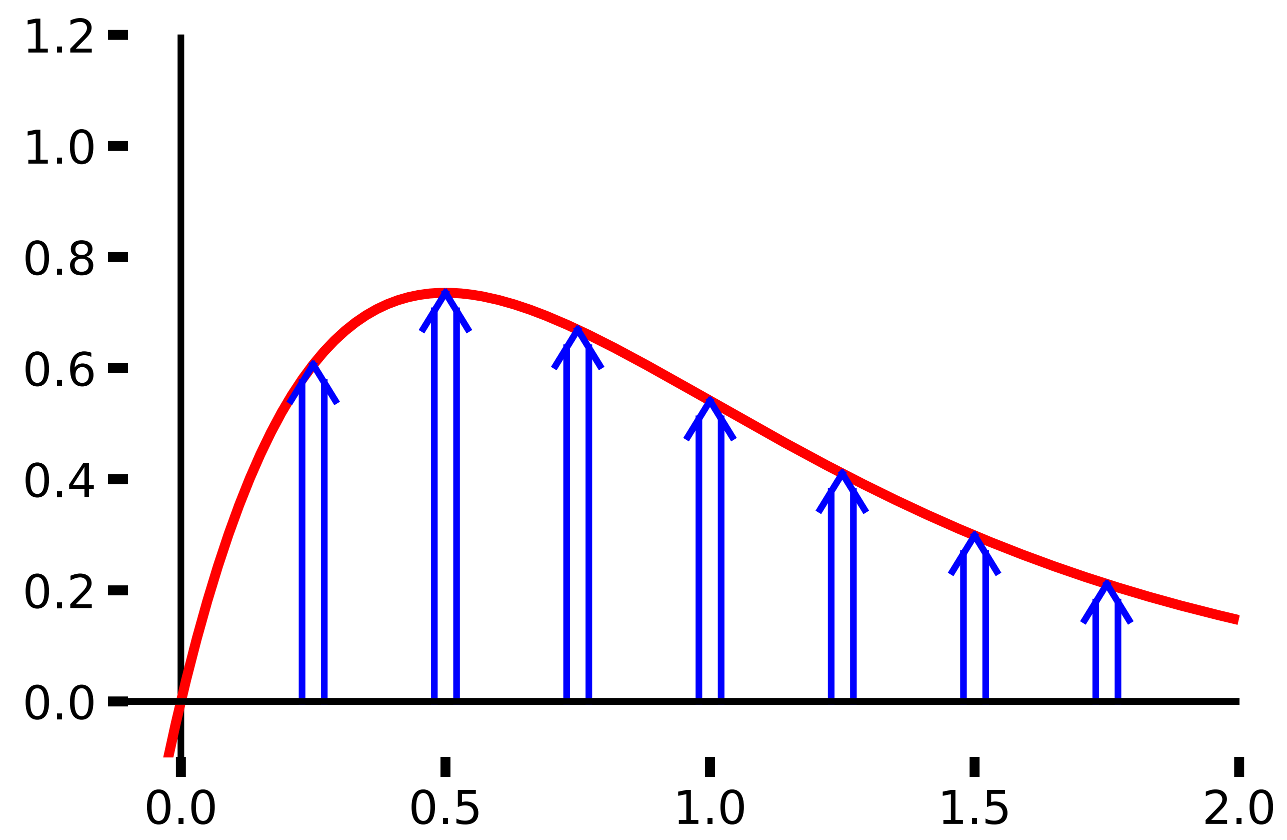
Impulse trains and sampling



Impulse trains can be used to model a sampled signal:

$$x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT_0) = \sum_{k=-\infty}^{\infty} x(kT_0)\delta(t - kT_0) \tag{8}$$

Impulse trains and sampling



Impulse trains can be used to model a sampled signal:

$$x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT_0) = \sum_{k=-\infty}^{\infty} x(kT_0)\delta(t - kT_0) \tag{8}$$