Linear Systems and Signals

Complex exponentials in DT

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Learning objectives

The learning objectives for this section are:

- Write the general form of complex exponential functions in DT.
- Use Euler's formula to express sinusoids in terms of complex exponentials.
- Sketch special cases of real exponentials, exponentially modulated sinusoids, and real sinusoids.
- Calculate fundamental period and frequency for a complex periodic exponential
- Explain similarities and differences between DT complex exponentials.

The DT complex exponential sequence

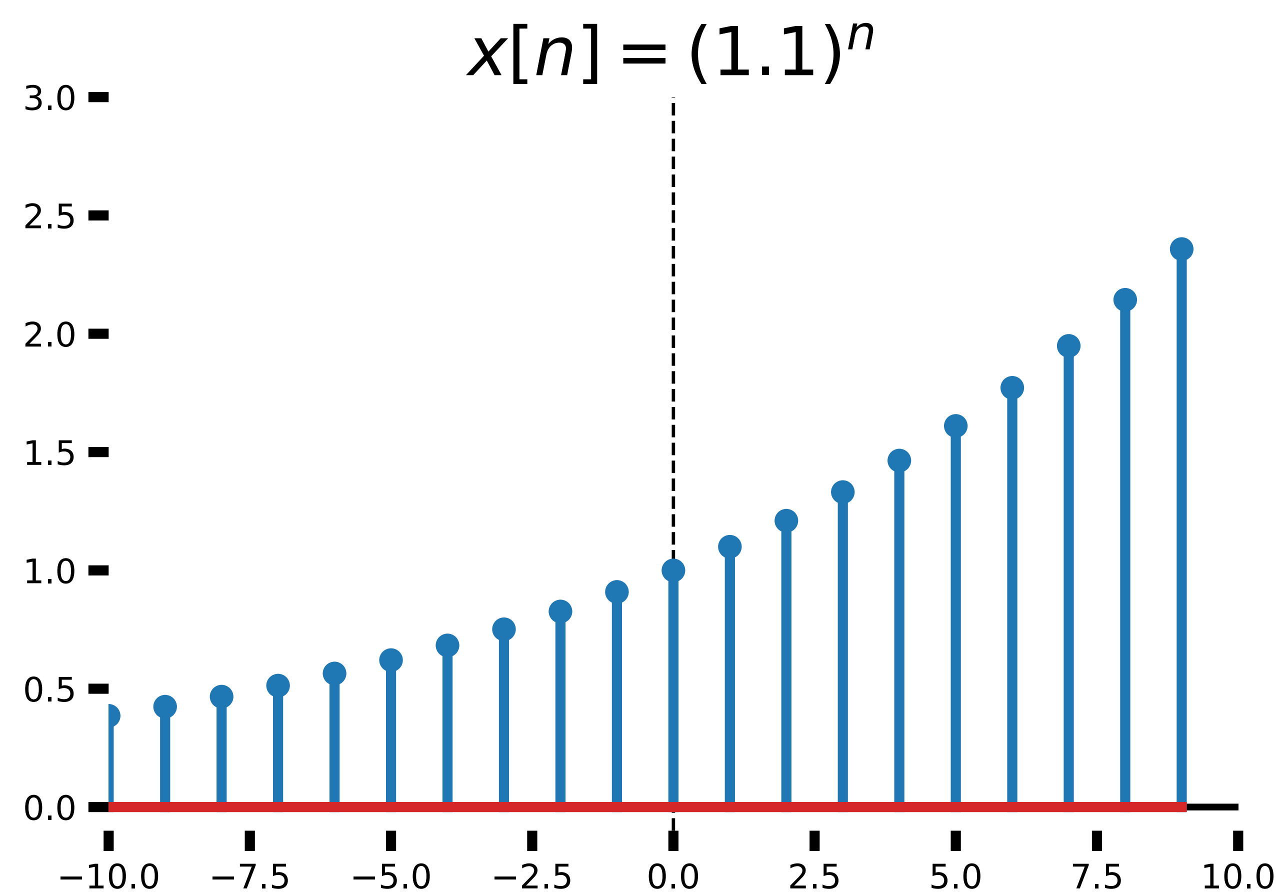
We have a *similar* story in DT. A complex exponential signal is of the form

$$x[n] = C\alpha^n \qquad C, \alpha \in \mathbb{C} \tag{1}$$

Why don't we write $\alpha = e^{\beta}$ where β is complex? Because it's often more convenient to write things in this form instead. That's because we will be doing a lot of geometric series:

$$\sum_{n=0}^{\infty} x[n] = C \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \qquad |\alpha| < 1 \tag{2}$$

Real signals



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$$x[n] = (1.1)^n \tag{3}$$

(4)

(5)

(6)

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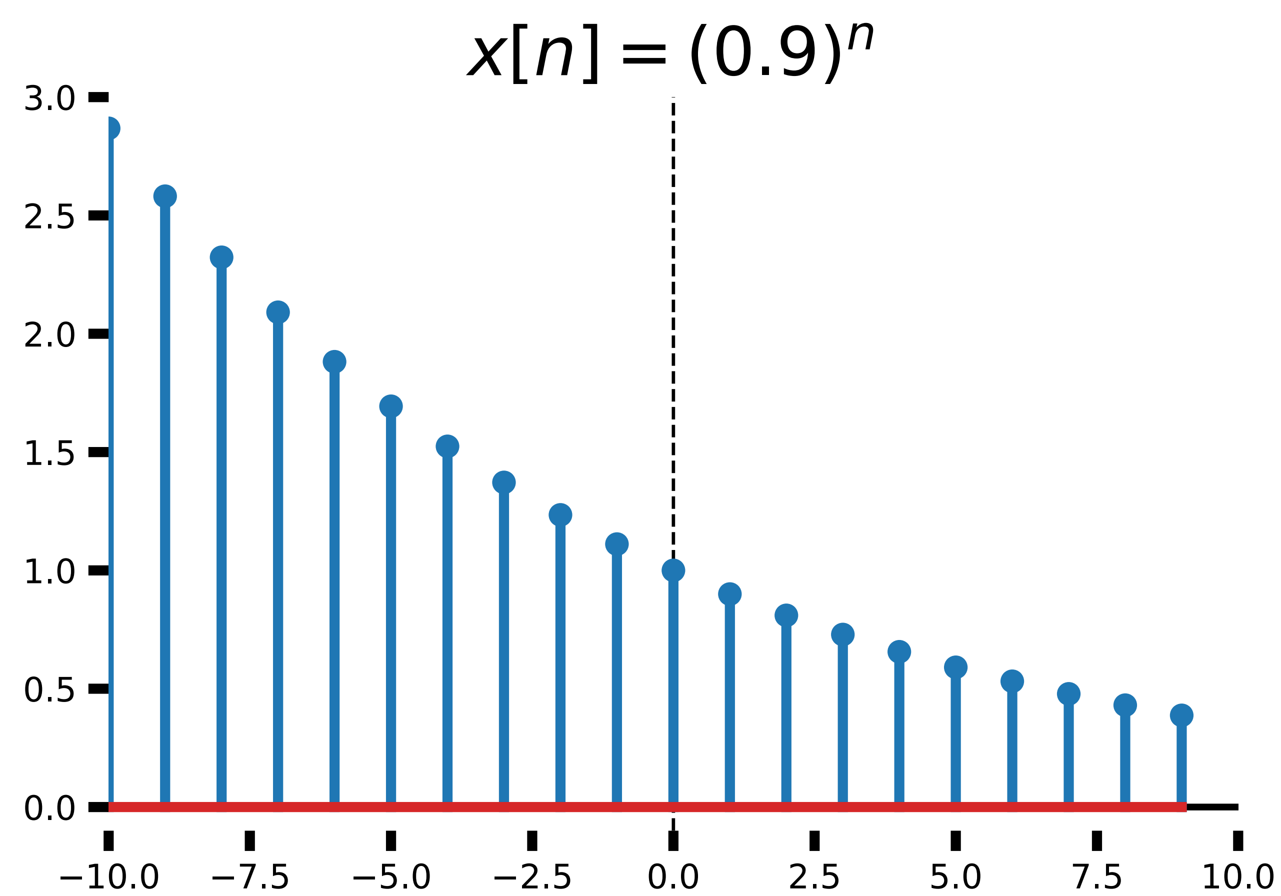

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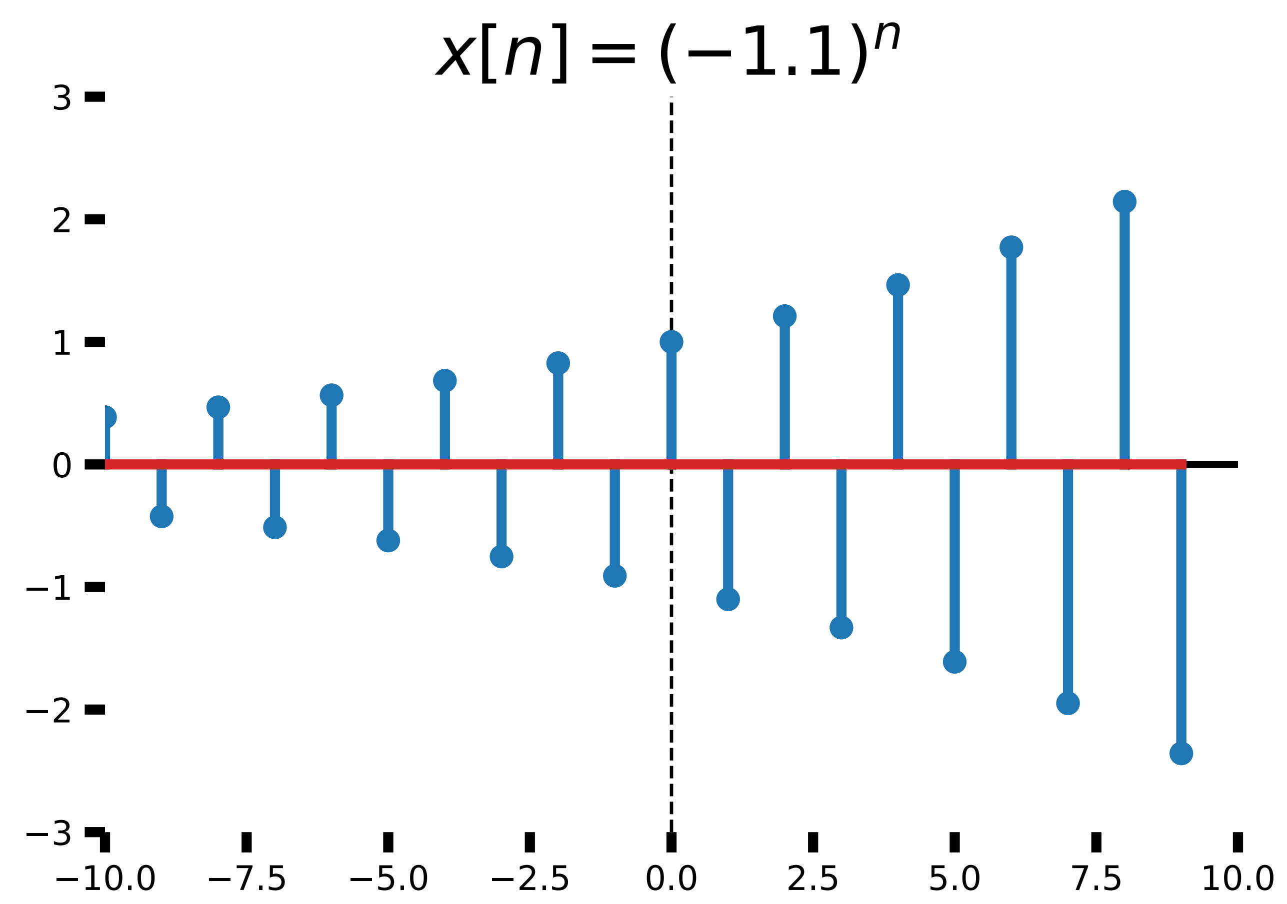

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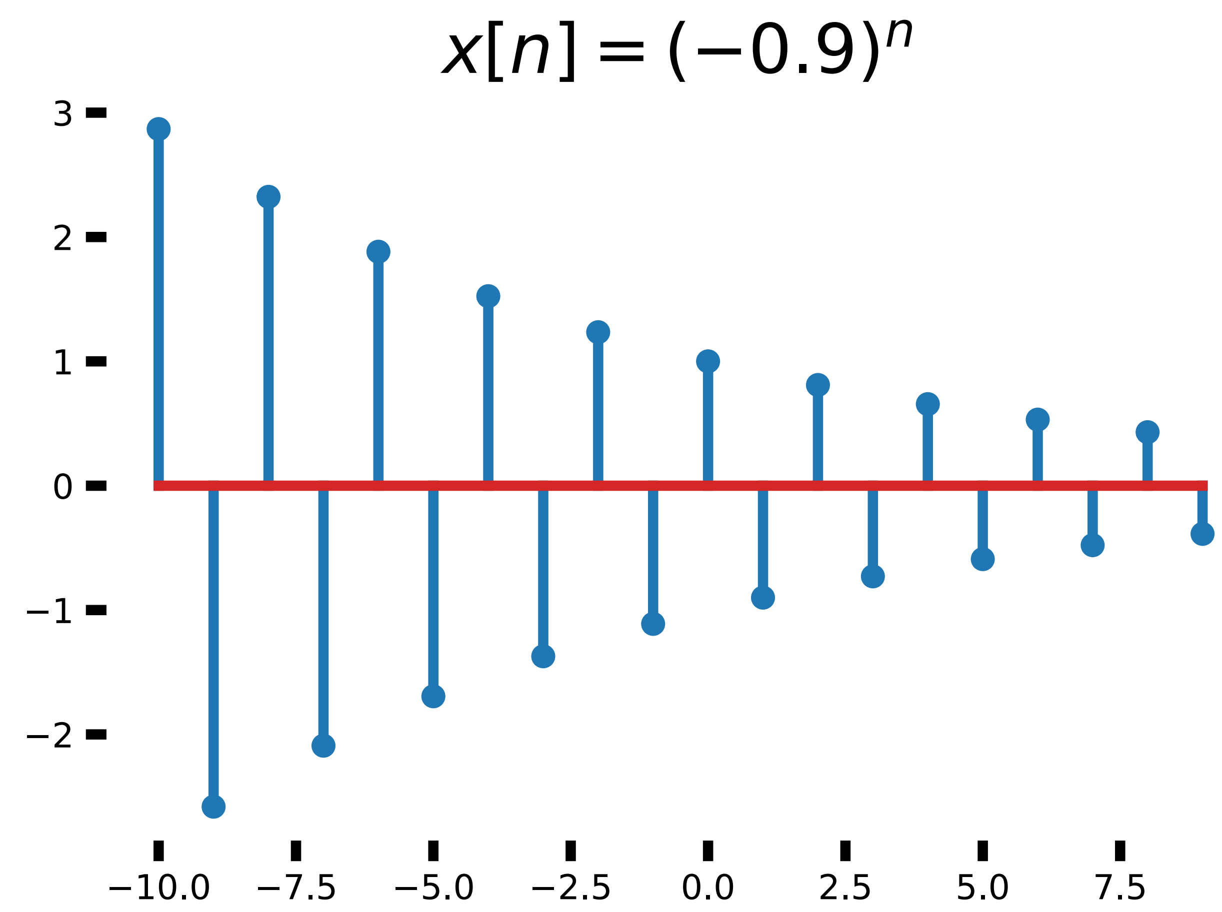

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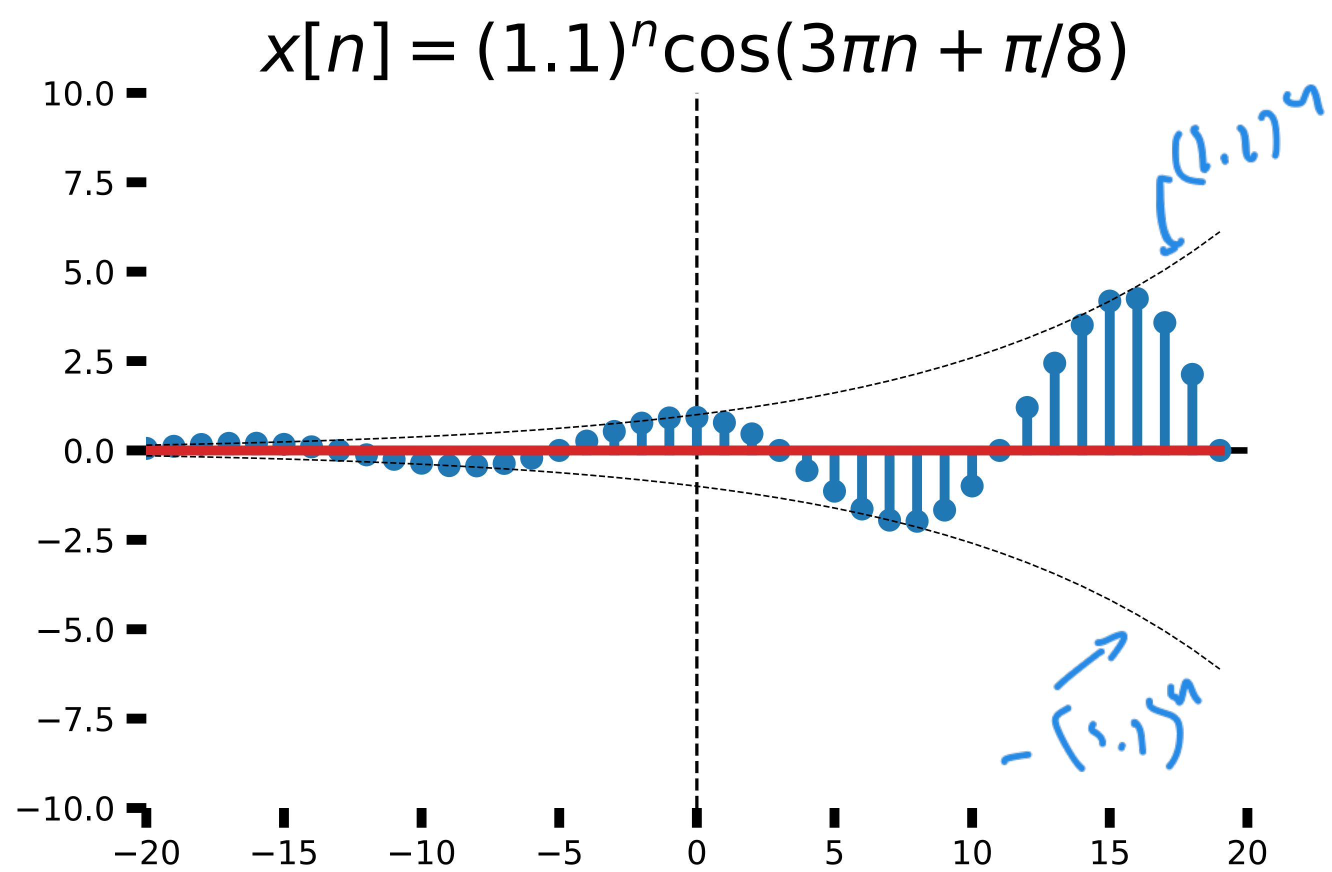
General complex exponentials



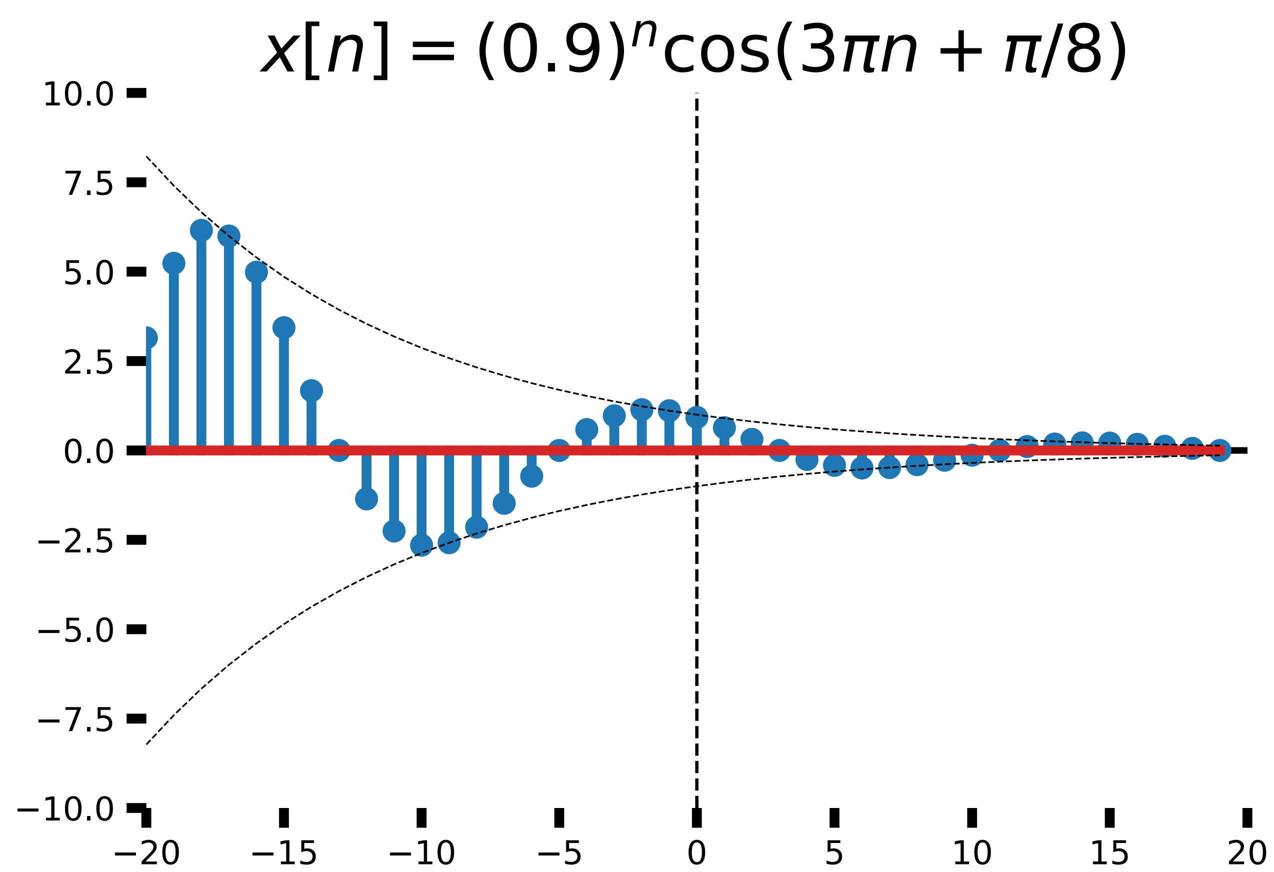
We can write $C = |C|e^{j\theta}$ and $\alpha = |\alpha|e^{j\omega_0}$ in magnitude-phase form to get

$$x[n] = |C||\alpha|^n e^{j(\omega_0 n + \theta)} \tag{7}$$

$$= |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta) \tag{8}$$

The real/imaginary part look similar to the CT version.

General complex exponentials



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Periodicity

Focusing on

$$x[n] = e^{j\omega_0 n} \tag{9}$$

we see that if we add/subtract any integer multiple of 2π to ω_0 we get the same thing:

$$e^{j(\omega_0+2\pi k)n} = e^{j\omega_0 n}e^{j2\pi k} = e^{j\omega_0 n}1^k = e^{j\omega_0 n}. \tag{10}$$

That means we can think of ω_0 as taking values in an interval of length 2π, such as $[0, 2\pi]$ or $[-\pi, \pi]$. For $x[n]$ to be *periodic* with period N we need

$$\omega_0 N = 2\pi m \qquad \Longrightarrow \qquad \frac{\omega_0}{2\pi} = \frac{m}{N}. \tag{11}$$

rational number

Periodicity

Not all DT complex exponentials are periodic! We have to have $\frac{2\pi}{\omega_0}$ be a rational number. The fundamental period of a complex exponential $e^{j\omega_0 n}$ is the smallest N such that there is an integer m with

$$N = m\left(\frac{2\pi}{\omega_0}\right). \tag{12}$$

Note, this is very different from CT, where $T_0 = \frac{2\pi}{|\omega_0|}$.

Example: (*OW p.29*) Suppose $x[n] = \cos(8\pi n/31)$ and $x(t) = \cos(8\pi t/31)$. Then

- $x(t)$ is periodic with period $\frac{2\pi}{8\pi/31} = \frac{31}{4}$.
- $x[n]$ is periodic with period 31.

Harmonically related exponentials

What about *harmonically related* exponentials in DT? The periodic complex exponential with period N has $\omega_0 = \frac{2\pi}{N}$. So

$$\phi_1[n] = e^{j(2\pi/N)n} \qquad (13)$$

$e^{j2\pi n} = 1$

$\sqrt[N]{1}$

The elements of $\phi_1[n]$ are also called the *N-th roots of unity*. We can define the *harmonically related* exponentials as those with integer multiples of ω_0:

$$\phi_k[n] = e^{jk(2\pi/N)n} \qquad (14)$$

Note: when k hits N we get $\phi_N[n] = e^{j2\pi n} = 1$ so $\phi_{N+1}[n] = \phi_1[n]$. If we define $\phi_0[n] = 1$ then there are only N unique harmonically related exponentials with period N.

An example

Take $N = 6$, so $\omega_0 = 2\pi/6 = \pi/3$:

n	0	1	2	3	4	5
$\phi_0[n]$	1	1	1	1	1	1
$\phi_1[n]$	1	$e^{j(\pi/3)n}$	$e^{j(2\pi/3)n}$	-1	$e^{j(4\pi/3)n}$	$e^{j(5\pi/3)n}$
$\phi_2[n]$	1	$e^{j(2\pi/3)n}$	$e^{j(4\pi/3)n}$	1	$e^{j(2\pi/3)n}$	$e^{j(4\pi/3)n}$
$\phi_3[n]$	1	-1	1	-1	1	-1
$\phi_4[n]$	1	$e^{j(4\pi/3)n}$	$e^{j(2\pi/3)n}$	1	$e^{j(4\pi/3)n}$	$e^{j(2\pi/3)n}$
$\phi_5[n]$	1	$e^{j(5\pi/3)n}$	$e^{j(4\pi/3)n}$	-1	$e^{j(2\pi/3)n}$	$e^{j(\pi/3)n}$

Here we make a few simplifications:

$$e^{j(3\pi/3)n} = -1 \tag{15}$$

$$e^{j(6\pi/3)n} = 1 \tag{16}$$

$$e^{j((m+6)\pi/3)n} = e^{j(m\pi/3)n} \tag{17}$$

Summary of differences between DT and CT

	$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
range of ω_0	$\omega_0 \in \mathbb{R}$	$\omega_0 \in [0, 2\pi]$
is periodic?	for any ω_0	for $\omega_0 = \frac{2\pi m}{N}$
fundamental frequency	ω_0	ω_0/m
fundamental period	$T_0 = \frac{2\pi}{\|\omega_0\|}$	$N = m\frac{2\pi}{\omega_0}$