Linear Systems and Signals

Conjugate symmetry

Anand D. Sarwate

Department of Electrical and Computer Engineering
Rutgers, The State University of New Jersey

2020

Learning objectives

The learning objectives for this section are:

- apply the definition of conjugate symmetry to complex signals

Symmetry for complex signals

The even and odd property are easier to think about for real signals. What about complex signals? We can still define even and odd in the same way:

even

$$x(t) = x(-t) \qquad x_{\text{even}}(t) = \frac{x(t) + x(-t)}{2} \tag{1}$$

odd

$$x(t) = -x(-t) \qquad x_{\text{odd}}(t) = \frac{x(t) - x(-t)}{2} \tag{2}$$

$$\tag{3}$$

But for complex signals we have one more type of symmetry called *conjugate symmetry*.

Conjugate symmetry

A signal is *conjugate symmetric* if

$$x(-t) = x^*(t) \qquad x[-n] = x^*[n]. \tag{4}$$

A signal is *conjugate antisymmetric* if

$$x(-t) = x^*(-t) \qquad x[-n] = x^*[-n]. \tag{5}$$

We will see the first one more often.

Conjugate symmetry and Cartesian representation

Suppose $x(t)$ is conjugate symmetric. Let's look at what happens to the real and imaginary parts:

$$x(t) = \mathfrak{Re}\{x(t)\} + j\mathfrak{Im}\{x(t)\} \tag{6}$$

$$x(-t) = \mathfrak{Re}\{x(-t)\} + j\mathfrak{Im}\{x(-t)\} \tag{7}$$

$$x^*(t) = \mathfrak{Re}\{x(t)\} - j\mathfrak{Im}\{x(t)\}. \tag{8}$$

Then since $x(-t) = x^*(t)$,

$$\mathfrak{Re}\{x(t)\} = \mathfrak{Re}\{x(-t)\} \tag{9}$$

$\Rightarrow$ real part is even!

$$\mathfrak{Im}\{x(t)\} = -\mathfrak{Im}\{x(-t)\}, \tag{10}$$

$\Rightarrow$ imag part is odd!

so the real part of $x(t)$ is even and the imaginary part is odd.

Conjugate symmetry and magnitude-phase representation

Suppose $x[n]$ is conjugate symmetric. Let's look at what happens to the magnitude and phase:

$$x[n] = |x[n]|e^{j\measuredangle x[n]} \tag{11}$$

$$x[-n] = |x[-n]|e^{j\measuredangle x[-n]} \tag{12}$$

$$x^*[n] = |x[n]|e^{-j\measuredangle x[n]}. \tag{13}$$

Then since $x[-n] = x^*[n]$,

$$|x[n]| = |x[-n]| \quad \to \text{even} \tag{14}$$

$$\measuredangle x[n] = -\measuredangle x[-n], \quad \to \text{odd} \tag{15}$$

so the magnitude of $x(t)$ is even and the phase is odd.

An example

Problem

Is the signal $X(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$ conjugate symmetric?

We can check the definition:

$$X(-\omega) = \frac{-1}{j\omega} + \pi\delta(\omega) = X^*(\omega) \tag{16}$$

so this signal is conjugate symmetric.

Note: we will see later that this signal is the *Fourier transform* of $u(t)$.

Try some examples

Problem

Check if the following signals are conjugate symmetric by applying the definition:

$$x_1(t) = e^{j200\pi t} \tag{17}$$

$$x_2(t) = \cos(4\pi t) + j\sin(3\pi t) \tag{18}$$

$$x_1[n] = e^{j\pi/3n} \tag{19}$$

$$x_2[n] = e^{j\pi/3n} - e^{j\pi/4n} \tag{20}$$