Linear Systems and Signals

Stable and unstable systems

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Learning objectives

The learning objective for this section is:

- determine if a system is stable or not

Stability

Definition

A system is *bounded-input bounded output (BIBO) stable* if for any input $x(t)$ (or $x[n]$) satisfying for all t (or n)

$$|x(t)| \leq B \qquad |x[n]| \leq B, \tag{1}$$

there is a function f such that $f(B) < \infty$ and

$$|\mathcal{H}(x(t))| \leq f(B) \qquad |\mathcal{H}(x[n])| \leq f(B). \tag{2}$$

That is, if the input is bounded for all time, the output is bounded for all time.

Stable and unstable systems

To show a system is stable you have to find the function $f(B)$.

- Suppose $y[n] = |x[n]|^2$, Then if $|x[n]| \leq B$ for all n, we have $|y[n]| \leq B^2$. So here $f(B) = B^2$.

- Suppose $y(t) = \int_{\tau=-\infty}^{t} x(\tau) d\tau$. This is an integrator. If we pick $x(t) = u(t)$ we can see that $y(t) = t$ which goes off to ∞ even though $|x(t)| \leq 1$.

A slightly more complex example

Consider

$$y(t) = \int_0^t x(\tau)\tau^4 e^{-2\tau}\cos(3\pi\tau)d\tau \tag{3}$$

Is this system stable?

First you should start with a guess. Let's guess that it is stable. Then we should show a bound on it. Suppose $\underline{|x(t)| < B}$. Then

$$|y(t)| = \left|\int_0^t x(\tau)\tau^4 e^{-2\tau}\cos(3\pi\tau)d\tau\right| \tag{4}$$

$$\leq \int_0^t \cancelto{B}{|x(\tau)|}\underline{|\tau^4||e^{-2\tau}|}\cancelto{1}{|\cos(3\pi\tau)|}d\tau \tag{5}$$

Now, $|x(\tau)| < B$ by assumption. The last part $|\cos(3\pi\tau)| \leq 1$ so we can upper bound it by 1. What about the the rest of it?

A slightly more complex example, continued

So far we have

$$|y(t)| \leq B \int_0^t |\tau^4||e^{-2\tau}| d\tau \tag{6}$$

Is the integral bounded? The exponential decays much faster than the polynomial term grows. So there's a T such that for $\tau > T$, we have $|\tau^4||e^{-2\tau}| < e^{-\tau}$ and for $\tau < T$, the quantity $|\tau^4||e^{-2\tau}| < T^4$. So the whole thing is bounded as

$$|y(t)| \leq B(T^5 + \int_T^t e^{-\tau} d\tau) \leq B(T^5 + 1). \tag{7}$$

What if you guessed wrong? You should see it not working out. Then go back and change your guess!

Showing a system is unstable

Is the following system stable?

$$y(t) = \int_0^\infty e^\tau x(t - \tau) d\tau. \tag{8}$$

We guess that the system is unstable since e^τ is blowing up. In fact, if we plug in $x(t) = u(t)$ we can see that the integral diverges.

What about integrating the output of the modulator system?

$$y(t) = \int_0^t x(\tau) \cos(6000\pi\tau) d\tau \tag{9}$$

Here $u(t)$ won't work but if you let $x(\tau) = \cos(6000\pi\tau)$ then you get the integral of $\cos^2(6000\pi\tau)$ which diverges.

Try it yourself

Problem

Determine if each of these systems is stable or unstable.

- $\mathcal{H}(x(t)) = \frac{d}{dt}x(t)$
- $\mathcal{H}(x[n]) = x_e[n]$
- $\mathcal{H}(x(t)) = x(t-2) + x(2-t)$
- $\mathcal{H}(x[n]) = nx[n]$