Linear Systems and Signals
Signals and vector spaces

Anand D. Sarwate

Department of Electrical and Computer Engineering
Rutgers, The State University of New Jersey

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Learning objectives

The learning objectives for this section are:

- interpret vector space properties for signals

Vector spaces

A vector space V is a set of objects called vectors along with two operations: scalar multiplication and vector addition. These satisfy 8 properties:

1. $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$.
2. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
3. There is a zero vector $\mathbf{0}$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for all $\mathbf{x} \in V$.
4. Every $\mathbf{x}$ has an additive inverse $-\mathbf{x}$ such that $\mathbf{x} + (-\mathbf{x}) = 0$.
5. $a(b\mathbf{x}) = (ab)\mathbf{x}$.
6. $1\mathbf{x} = \mathbf{x}$.
7. $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + b\mathbf{y}$.
8. $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$.

Signals are vectors

You've already seen vector spaces like $\mathbb{R}^n$ and $\mathbb{C}^n$. We saw already that we represent signals as vectors computationally, but they are actually vectors in the *vector space of functions*. Why make a big deal out of this?

- When we think about vectors in $\mathbb{R}^n$ and $\mathbb{C}^n$ we think about them *geometrically*.
- We will mostly focus on linear systems, which correspond to *linear transformations* on these vector spaces.
- When we think of vector spaces we think of basis vectors. It turns our that Fourier transforms are really just a *change of basis*.

Function spaces

When we talk about signals as vectors we are thinking of them as functions in a vector space of functions, or a *function space*.

- We can think of complex CT functions with finite energy:

$$\mathcal{L}_2(\mathbb{R}) = \left\{ x(t) : \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty \right\} \tag{1}$$

- We can also think of complex DT functions with finite energy:

$$\ell_2(\mathbb{Z}) = \left\{ x[n] : \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \right\} \tag{2}$$

If we think about signals in $\mathcal{L}_2$ and ℓ_2 they satisfy the axioms (try checking them for yourself: is the sum of two finite-energy signals also finite-energy?)

The inner product

We actually have even more structure in the space of signals: we have an inner product, which in $\mathbb{R}^n$ is just a fancy name for a dot product:

$$\langle x(\cdot),\ y(\cdot)\rangle = \int_{-\infty}^{\infty} x(t)y^*(t)dt \tag{3}$$

$$\langle x[\cdot],\ y[\cdot]\rangle = \sum_{k=-\infty}^{\infty} x[n]y^*[n] \tag{4}$$

These conjugates mean $\langle x,\ y\rangle = \langle y,\ x\rangle^*$.

Norms

In $\mathbb{R}^n$ we have $\langle \mathbf{x}, \ \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{y}$, which means $\langle \mathbf{x}, \ \mathbf{x} \rangle = \|\mathbf{x}\|^2$, the *squared length of* $\mathbf{x}$.

Similarly, the *squared norm* of a signal is

$$\|x(\cdot)\| = \int_{-\infty}^{\infty} x(t)x^*(t)dt = \int_{-\infty}^{\infty} |x(t)|^2 dt = \mathcal{E}_x \tag{5}$$

$$\|x[n]\| = \sum_{k=-\infty}^{\infty} x[n]x^*[n] \sum_{k=-\infty}^{\infty} |x[n]|^2 = \mathcal{E}_x \tag{6}$$

$\langle x, x \rangle$

That is, the *energy* of a signal is the *squared length* of that signal *when we think of it as a vector.*

Projections

In $\mathbb{R}^n$ we think of $\mathbf{x}^\top \mathbf{y}$ as the *component of* $\mathbf{x}$ *in the direction of* $\mathbf{y}$ or the *projection of* $\mathbf{x}$ *onto* $\mathbf{y}$. The same thing is true for function/signal spaces with an inner product.

We will see later that the Fourier transform of $x(t)$ or $x[n]$ can be thought of as an inner product of x with a complex exponential.[1]

[1] Note that $e^{j\omega t} \neq \mathcal{L}_2(\mathbb{R})$ and $e^{j\omega n} \neq \ell_2(\mathbb{Z})$, so the story is a little different.

Linear transformations

In most of this class we will be interested in linear transformations on signals. For finite-length vectors in $\mathbb{R}^n$ we can think of a linear transformation as multiplication by a matrix:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{7}$$

Can we do the same interpretation for signals?

DT transformations: intuition

For DT signals we can think of this as "multiplication" by an "infinitely large" matrix:

$$\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & a_{r,1} & a_{r,2} & \cdots & a_{r,n} & \cdots \\ \cdots & a_{r+1,1} & a_{r+1,2} & \cdots & a_{r+1,n} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \\ \cdots & a_{r+n,1} & a_{r+n,2} & \cdots & a_{r+n,n} & \cdots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ x_r \\ x_{r+1} \\ \vdots \\ x_{r+n} \\ \vdots \end{bmatrix} \tag{8}$$

For CT signals the matrix analogy doesn't work as well.

DT transformations: delay by 1

Consider a delay by 1: this is just shifting the vector down by one position:

$Gx = x$ for all x $G =$ Identity Matrix

eye(n)

$$\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & \ddots & 1 \end{bmatrix}$$

$$\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \ddots \\ \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots \\ \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots \\ \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ x_r \\ x_{r+1} \\ x_{r+2} \\ \vdots \\ x_{r+n} \\ x_{r+n+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ x_{r+1} \\ x_{r+2} \\ x_{r+3} \\ \vdots \\ x_{r+n+1} \\ x_{r+n+2} \\ \vdots \end{bmatrix}. \quad (9)$$

delay

$$\begin{bmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & 0 & 1 \\ 0 & & & \ddots \end{bmatrix}$$

Summary and things to try

The vector space view of signals is very useful:

- Inner products are projections.
- Energy is the squared length of the signal as a vector.
- Linear transformations can look like matrix multiiplication.

Try computing the inner product between signals like $\cos(\omega t)$ and $\sin(\omega t)$. Try it with complex signals too!