Linear Systems and Signals

Complex exponentials in CT

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Learning objectives

The learning objectives for this section are:

- Write the general form of complex exponential functions in CT.
- Use Euler's formula to express sinusoids in terms of complex exponentials.
- Sketch special cases of real exponentials, exponentially modulated sinusoids, and real sinusoids.
- Calculate fundamental period and frequency for a complex periodic exponential
- Explain similarities and differences between CT complex exponentials.

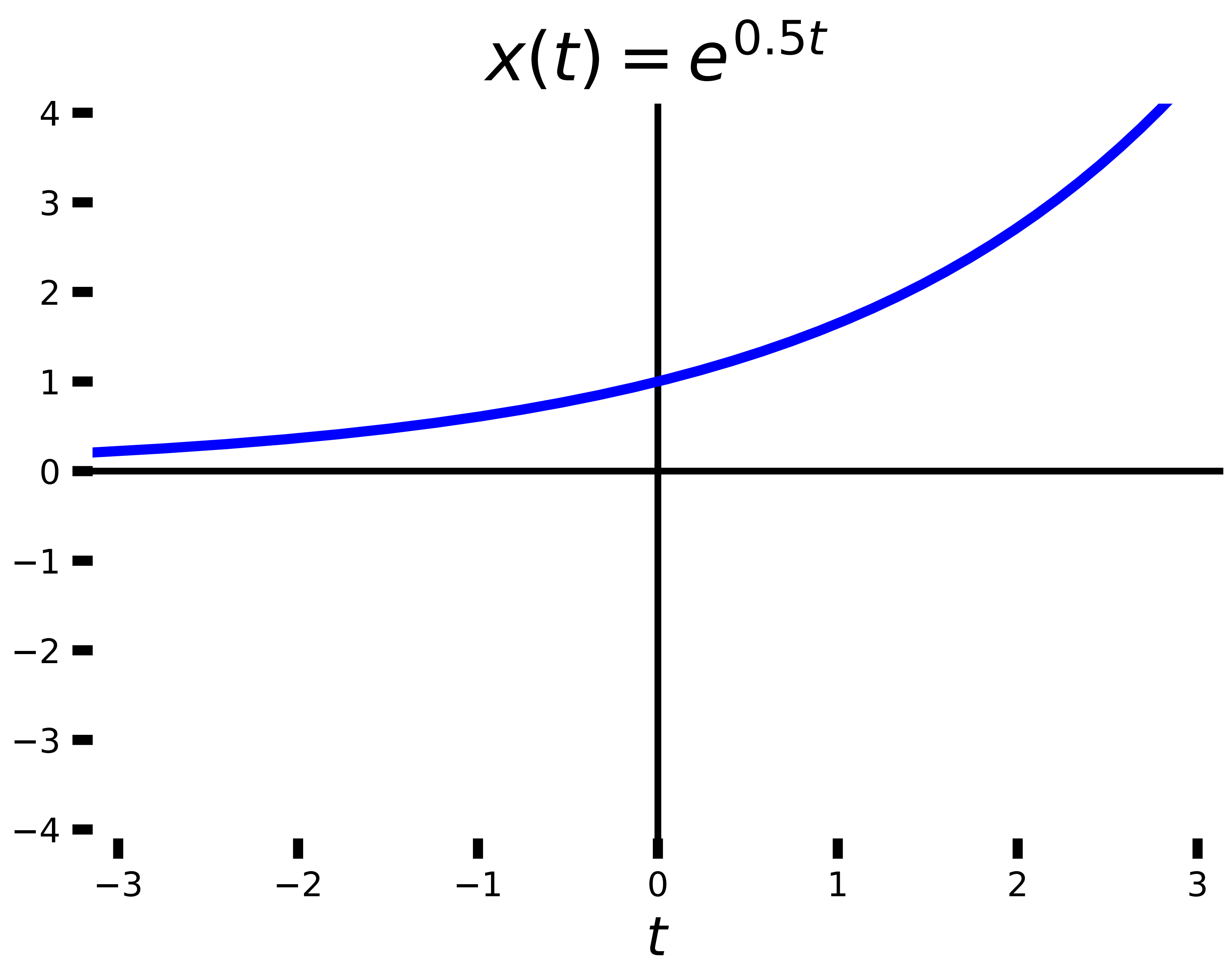
Basic definition

A CT *complex exponential signal* is of the form

$$x(t) = Ce^{at} \tag{1}$$

where C and a are complex numbers. All the other things we will see will be special cases of this form.

Real signals



If $x(t) = Ce^{at}$ is a *real* signal, then C and a are real numbers. These should be pretty familiar looking: if $a > 0$ then there is an exponential increase and if $a < 0$ there is an exponential decrease.

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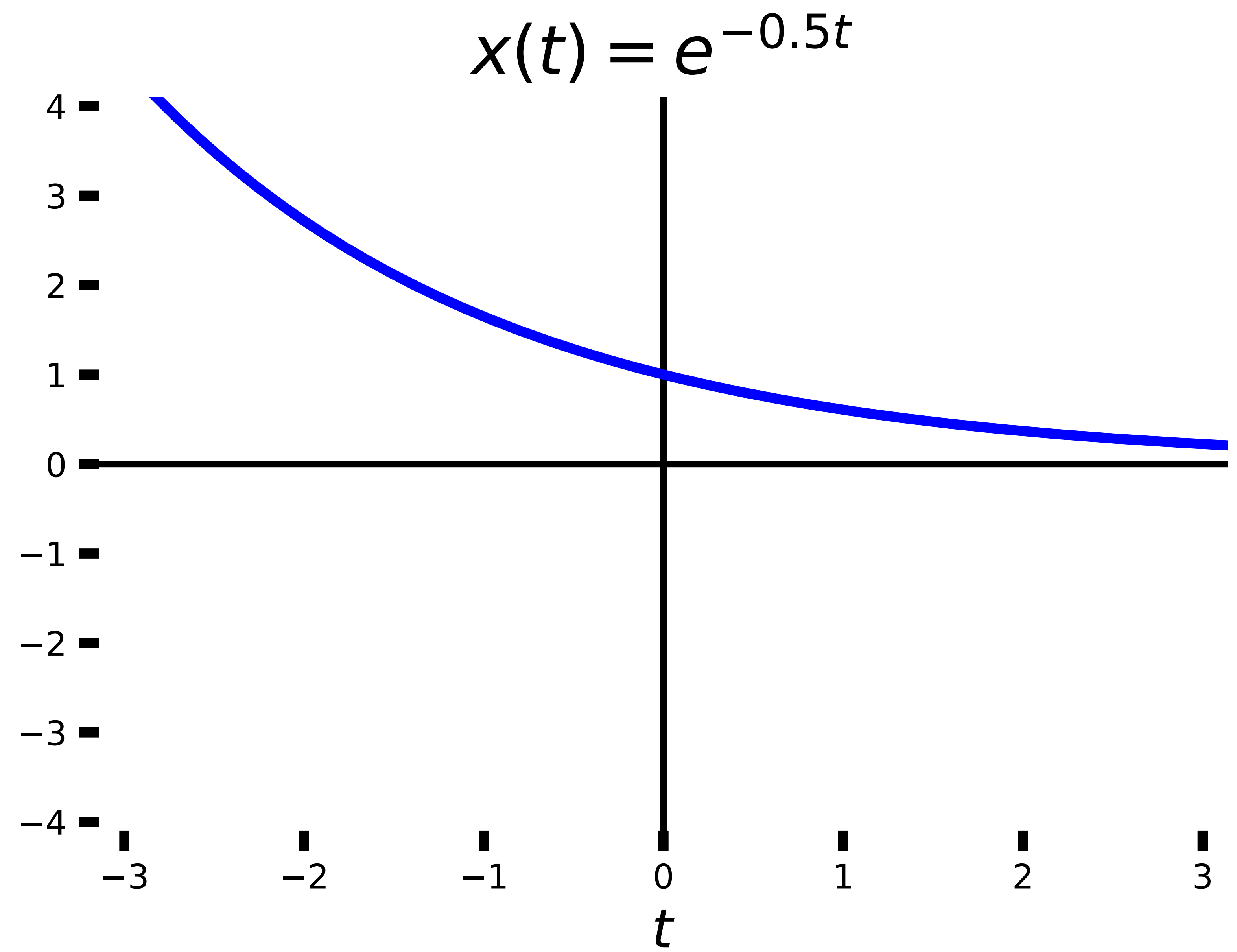

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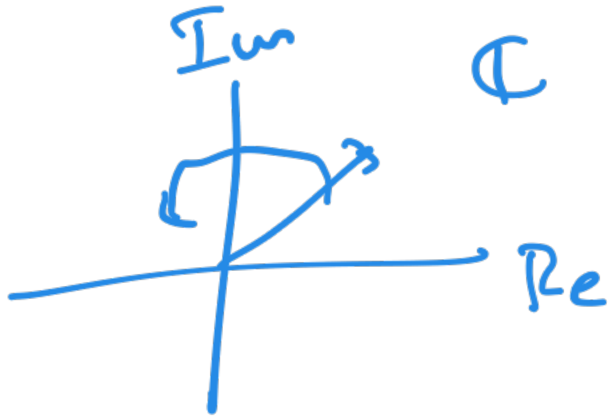
Complex signals

If we make a purely imaginary then we have

$$x(t) = Ce^{j\omega_0 t} = |C|e^{j(\omega_0 t + \measuredangle C)} \tag{2}$$

Let's focus on the simple case where $C = 1$. Then we have

$$x(t) = e^{j\omega_0 t}. \tag{3}$$



This is *periodic* with fundamental period

$$T_0 = \frac{2\pi}{|\omega_0|}. \tag{4}$$

Eulerizing and sinusoids

Euler's formula is going to be used *a lot* in this course:

$$e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t) \tag{5}$$

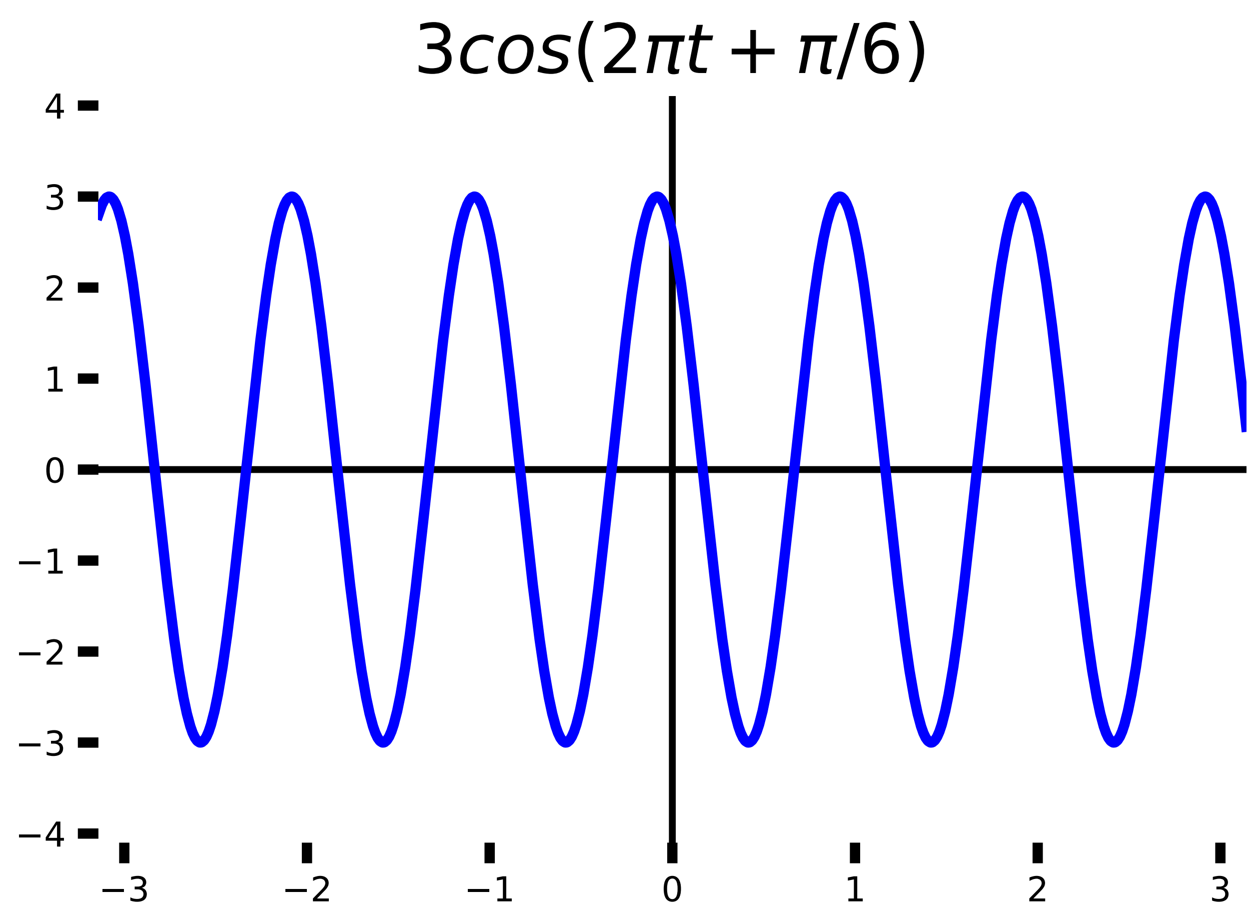
What this implies is that regular sines and cosines are linear combinations of complex exponentials:

$$\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \tag{6}$$

$$\sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \qquad \frac{1}{j} = -j \tag{7}$$

It's often very useful to "Eulerize" sines and cosines so that you can "gather like terms" when doing algebra. As an exercise, try getting a formula for $\cos(\omega_0 t)\sin(\omega_0 t)$ by Eulerizing, multiplying, and then simplifying back to real signals.

Now with phases!



If we have a phase shift:

$$A\cos(\omega_0 t + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 t} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 t} = A\,\mathfrak{Re}\left\{e^{j\omega_0 t + \phi}\right\} \quad (8)$$

Note that the amplitudes of each term are complex-valued (in our general formula, C is complex). Likewise:

$$A\sin(\omega_0 t + \phi) = A\,\mathfrak{Im}\left\{e^{j\omega_0 t + \phi}\right\}. \quad (9)$$

Harmonically related sinusoids

If we have a complex exponential $\phi_1(t) = e^{j\omega_0 t}$ with period $T_0 = \frac{2\pi}{|\omega_0|}$ we can define the *harmonically related* complex exponentials, which are

$$\phi_k(t) = e^{jk\omega_0 t} \qquad k = 0, \pm 1, \pm 2, \dots \tag{10}$$

These are all of the periodic exponentials whose frequency is an integer multiple of $\phi_1(t)$.

This terminology comes from music (and the physics thereof): a vibrating string or standing wave is the superposition (linear combination) of harmonically related exponentials.

The general case

More generally, we can split C into its magnitude and phase form and a into its Cartesian form:

$$C = |C|e^{j\theta} \qquad\qquad a = r + j\omega_0 \tag{11}$$

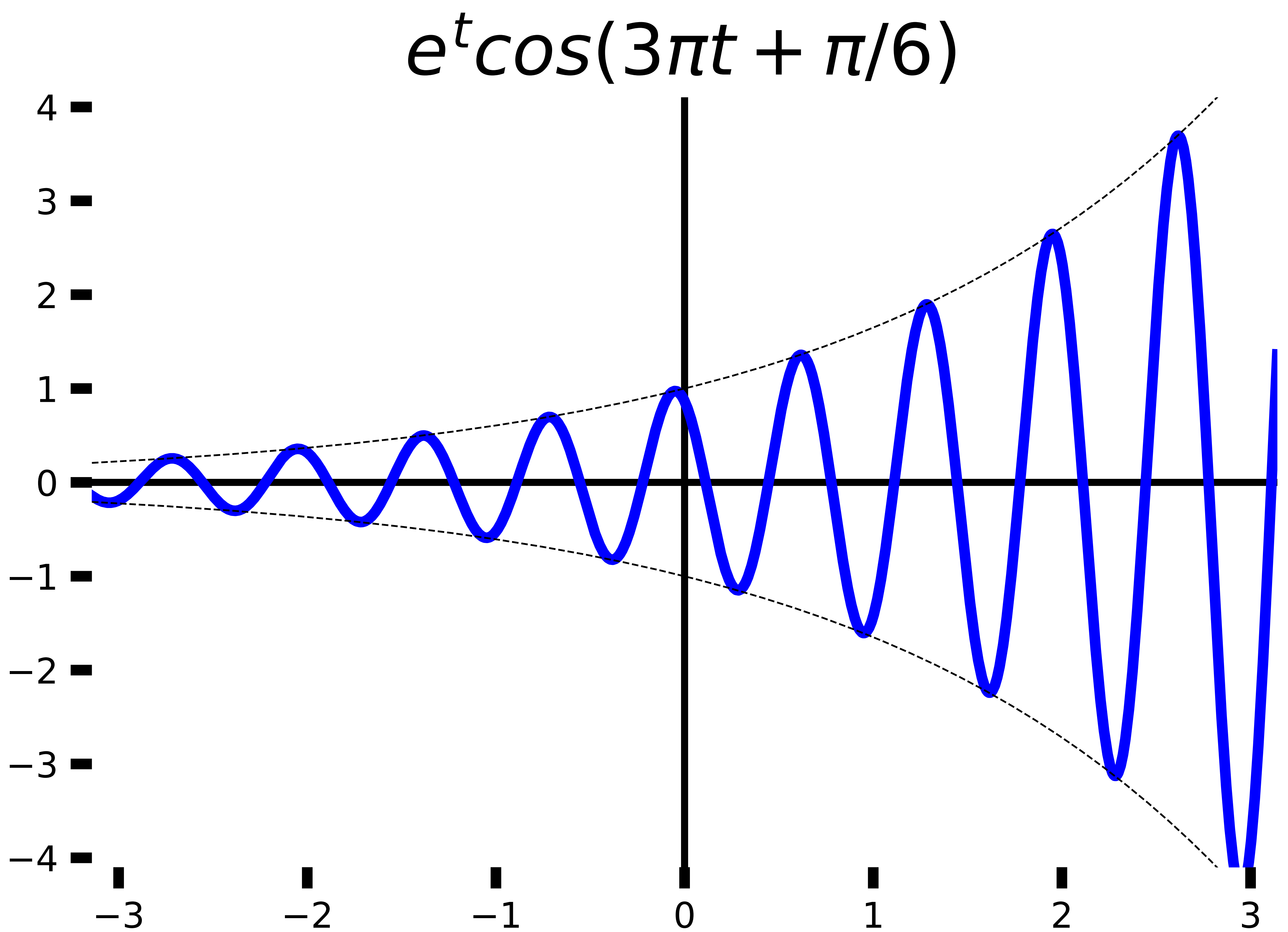
and look at functions of the form

$$x(t) = Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t} = |C|e^{rt}\underline{e^{j(\omega_0 t+\theta)}} \tag{12}$$

Eulerizing:

$$x(t) = |C|e^{rt}\underline{\cos(\omega_0 t + \theta)} + j|C|e^{rt}\underline{\sin(\omega_0 t + \theta)} \tag{13}$$

The general case in pictures



The general case in pictures