ECE 345: Linear Systems and Signals — Fall 2020
Professors Sarwate and El Rouayheb — Due: 5 PM, September 25, 2020

Solutions for Homework 2

Problem 1 (SSTA Problem 2.23)**.** Determine whether or not each of the following LTI systems is: (1) causal and/or BIBO-stable. If the system is not BIBO stable, provide a bounded input that yields an unbounded output.

(a) $y(t) = \frac{dx}{dt}$

(b) $y(t) = \int_{-\infty}^{t} x(\tau) d\tau$

(c) $y(t) = \int_{-\infty}^{t} x(\tau)\cos(t-\tau) d\tau$

(d) $y(t) = x(t+1)$

(e) $y(t) = \int_{t-1}^{t+1} x(\tau) d\tau$

(f) $y(t) = \int_{t}^{\infty} x(\tau) e^{2(t-\tau)} d\tau$

Solution:

To show a system is causal, we need to look at what values of $x(\cdot)$ are needed to compute $y(t)$ at time t. To show a system is stable we need to show that if $|x(t)| \leq B$ then the output will satisfy $|y(t)| \leq f(B)$. To show it is unstable we only need to find some example input signal $x(t)$ which will cause the output to be unbounded.

(a) This system is causal since it depends only on $\{x(\tau), \tau \leq t\}$. It is not BIBO-stable since with input $x(t) = u(t)$ we get output $y(t) = \delta(t)$, which is unbounded. This illustrates the key concept that the derivative of the unit step is the delta function.

(b) This system is causal since $y(t)$ depends only on $\{x(\tau) : \tau \leq t\}$. It is not BIBO-stable since a bounded $x(t) = u(t)$ results in $y(t) = tu(t)$, which is unbounded (plot it!).

(c) This system is causal since $y(t)$ depends only on $\{x(\tau) : \tau \leq t\}$. It is not BIBO-stable: we can input $x(t) = \cos(t)$ to get $y(t)$ unbounded.

(d) This system is not causal since $y(t)$ depends on $x(t+1)$, which is in the future. It is BIBO-stable since if $|x(t)| \leq B$ then $|y(t)| = |x(t+1)| \leq B$.

(e) This system is not causal since $y(t)$ depends on future input values $\{x(\tau) : t < \tau < t+1\}$. It is BIBO-stable: if $|x(t)| \leq M$ then $|y(t)| \leq 2M$. The length of the interval of integration is $(t+1) - (t-1) = 2$.

(f) This system is not causal since since $y(t)$ depends on future $\{x(\tau) : t < \tau\}$. It is BIBO-stable: this is just the time-reversal of a system with impulse response $e^{-2t}u(t)$. However, since we haven't seen CT impulse responses, we can see this by upper bounding $e^{2(t-\tau)} \leq 1$ for $\tau > t$.

Problem 2 (Time-invariance)**.** Determine whether the following systems are time-invariant or time-varying. Note the first two are CT and the second two are DT.

(a) $y(t) = \sin(x(t))$

(b) $y(t) = t\sin(x(t))$

(c) $y[k] = 3(x[k] - x[k-2])$

(d) $y[k] = kx[k]$

Solution:

(a) Time-invariant. You can check the definition: delaying $x(t)$ by τ to get $x(t-\tau)$ produces $y(t-\tau)$.

(b) Time-varying. Here delaying $x(t)$ by τ does not delay the multiplicative t on the outside of the sine by τ, so $z(t) = t\sin(x(t-\tau)) \neq y(t-\tau) = (t-\tau)\sin(x(t-\tau))$.

(c) Time-invariant. This is actually an LTI system.

(d) Time-varying. Much like the in CT case above, the presence of the k makes things time-invariant: $z[k] = kx[k-m] \neq y[k-m] = (k-m)x[k-m]$.

Problem 3 (Even and odd parts)**.** Express the following signal as the sum of an even and an odd signal

$$x(t) = \begin{cases} t & 0 \leq t < 1 \\ 0 & \text{elsewhere} \end{cases} \tag{1}$$

Plot the even and odd parts.

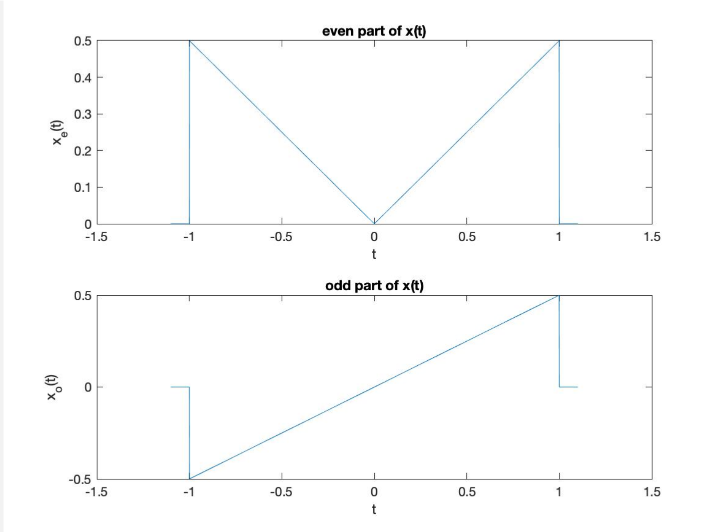
Solution:

The even and odd parts are

$$x_{\text{even}}(t) = \frac{x(t) + x(-t)}{2} = x(t) = \begin{cases} |t|/2 & -1 < t < 1 \\ 0 & \text{elsewhere} \end{cases} \tag{2}$$

$$x_{\text{odd}}(t) = \frac{x(t) - x(-t)}{2} = x(t) = \begin{cases} t/2 & -1 < t < 1 \\ 0 & \text{elsewhere} \end{cases} \tag{3}$$

They are not causal. Here are some plots:



Problem 4 (ECE 345 Final Exam, Fall 2017)**.** Consider the system $y[n] = x_{\text{even}}[n]$ that outputs the even part of a function. Determine whether or not this system is memoryless, time-invariant, linear, or stable.

Solution:

We have the formula

$$x_{\text{even}}[n] = \frac{1}{2}(x[n]) + x[-n]). \tag{4}$$

From this we can see that this system is not memoryless since the output at time k depends on $x[k]$ and $x[-k]$.

Now suppose $x[n] = u[n+1] - 2u[n] + u[n-1]$, which is an odd function from -1 to 1. This is an odd signal so $y[n] = 0$. But $x[n]$ delayed by 1 is $u[n-2] - 2u[n-1] + u[n]$ which is neither even nor odd so the output is not $y[n-1] = 0$. So it is not time-invariant.

Suppose $x[n] = a_1 x_1[n] + a_2 x_2[n]$. Then

$$\begin{aligned} x_{\text{even}}[n] &= \frac{1}{2}(a_1 x_1[n] + a_2 x_2[n] + a_1 x_1[-n] - a_2 x_2[-n]) && (5) \\ &= a_1 \frac{1}{2}(x_{1,\text{even}}[n] + x_{1,\text{even}}[-n]) + a_2 \frac{1}{2}(x_{2,\text{even}}[n] + x_{2,\text{even}}[-n]) && (6) \\ &= a_1 x_{1,\text{even}}[n] + a_2 x_{2,\text{even}}[n] && (7) \end{aligned}$$

So this system is linear.

If $|x[n]| \le B$ for all t then $|y(t)| \le \frac{1}{2}(|x[n]| + |x[-n]|) \le B$ as well, so the system is stable.

Problem 5 (SSTA Problem 2.23)**.** Consider the CT system with the following input-output relation:

$$y(t) = x(t)\cos(120\pi t) + x(t-3). \tag{8}$$

Determine if this system is stable/unstable, causal/noncausal, linear/nonlinear, and time-invariant or time-varying.

Solution:

This system is stable since if $|x(t)| < B$ then $|y(t)| < |x(t)||\cos(120\pi t)| + |x(t-3)| < 2B$.

The system is causal since $y(t)$ only depends on $x(t)$ and $x(t-3)$, which is in the past.

The system is linear because

$$(\alpha_1 x_1(t) + \alpha_2 x_2(t))\cos(120\pi t) + \alpha_1 x_1(t-3) + \alpha_2 x_2(t-3) \tag{9}$$
$$= \alpha_1(x_1(t)\cos(120\pi t) + x_1(t-3)) + \alpha_2(x_2(t)\cos(120\pi t) + x_2(t-3)) \tag{10}$$
$$= y_1(t) + y_2(t). \tag{11}$$

This system is time varying because delaying $x(t)$ does not delay the cosine, so

$$y(t - 1/240) \tag{12}$$
$$= x(t-1/240)\cos(120\pi(t-1/240)) + x(t-3-1/240) \tag{13}$$
$$= x(t-1/240)\cos(120\pi t - \pi/2) + x(t-3-1/240) \tag{14}$$
$$= x(t-1/240)\sin(120\pi t) + x(t-3-1/240) \tag{15}$$
$$\neq x(t-1/240)\cos(120\pi t) + x(t-3-1/240) \tag{16}$$

This might be tricky if you only think about integer t: for $t \in \mathbb{Z}$ the cosine is always 1 so it effectively goes away. But for CT it is not always 1.

Problem 6 (Lathi and Green 1.7-16)**.** For the systems described by the following equations, with input $x(t)$ and output $y(t)$, determine which are invertible and which are noninvertible. For the invertible systems, find the input-output relationship of the inverse system.

(a) $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$

(b) $y(t) = \frac{dx(t)}{dt}$ for differentiable $x(t)$.

(c) $y(t) = x(3t-6)$

(d) $y(t) = \cos(x(t))$

Solution:

(a) By the fundamental theorem of calculus, we know $x(t) = \frac{d}{dt}y(t)$ so this system is invertible.

(b) This question is more subtle than anticipated since you need to know the initial conditions to invert the derivative with an integral. But without any initial conditions the integral $\int_{-\infty}^{t} y(\tau)d\tau$ is the inverse system.

(c) We have $y(1) = x(-3)$, $y(2) = x(0)$ and $y(3) = x(3)$. Solving for $\tau = 3t - 6$ means $t = \frac{1}{3}\tau + 2$, so $x(t) = y(\frac{1}{3}t + 2)$.

(d) The cosine function is not one-to-one so this system is not invertible.

Problem 7 (ECE 345 Final Exam, Fall 2017)**.** Consider the upsampler by K:

$$y[n] = \begin{cases} x[n/K] & n/K \text{ is an integer} \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

Is the upsampler system linear? Either prove that the upsampler is linear or provide a counterexample to show that it is not linear.

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Solution:

It is a linear system:

$$a_1x_1[n] + a_2x_2[n] \to \begin{cases} a_1x_1[n/K] + a_2x_2[n/K] & n/K \text{ is an integer} \\ 0 & \text{otherwise} \end{cases} \tag{18}$$
$$= a_1y_1[n] + a_2y_2[n] \tag{19}$$

Problem 8. For the following two DT LTI systems, find the impulse response and the output of the system with input

$$x[n] = 2\delta[n] + 3\delta[n-1] \tag{20}$$

(a) $y[n] = x[n-1] + 2x[n-3]$

(b) $y[n+1] - 0.4y[n] = x[n]$

Solution:

(a) For this one, the impulse response is simple since there is no feedback term:

$$h[n] = \delta[n-1] + 2\delta[n-3] \tag{21}$$

We can do the convolution manually by making delayed "copies" of $x[n] = 2\delta[n] + 3\delta[n-1]$:

$$\begin{aligned} y[n] &= 2h[n] + 3h[n-1] & (22)\\ &= 2(\delta[n-1] + 2\delta[n-3]) + 3(\delta[n-2] + 2\delta[n-4]) & (23)\\ &= 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3] + 6\delta[n-4]. & (24) \end{aligned}$$

(b) This system has feedback so we need to expand out the recursion:

$$\begin{aligned} y[n] &= x[n-1] + 0.4y[n-1] & (25)\\ &= x[n-1] + 0.4x[n-2] + (0.4)^2y[n-2] & (26)\\ &= x[n-1] + 0.4x[n-2] + (0.4)^2x[n-3] + (0.4)^3y[n-4] & (27)\\ &= \sum_{k=1}^{\infty}(0.4)^{k-1}x[n-k] & (28) \end{aligned}$$

Therefore

$$h[n] = \sum_{k=1}^{\infty}(0.4)^{k-1}\delta[n-k]. \tag{29}$$

The output with input $x[n] = 2\delta[n] + 3\delta[n-1]$ can be found by making "copies" of $h[n]$:

$$\begin{aligned} y[n] &= 2h[n] + 3h[n-1] & (30)\\ &= 2\sum_{k=1}^{\infty}(0.4)^{k-1}\delta[n-k] + 3\sum_{\ell=1}^{\infty}(0.4)^{\ell-1}\delta[n-\ell-1]. & (31) \end{aligned}$$

Problem 9. Consider the DT LTI system with impulse response

$$h[n] = \left(\frac{1}{2}\right)^n u[n] \tag{32}$$

What is the output of the system to the following inputs?

(a) $x[n] = \delta[n-1] + 2\delta[n-3] + 2\delta[n-5]$

(b) $x[n] = \left(\frac{1}{3}\right)^n u[n-3]$

Solution:

(a) This signal $x[n]$ makes three copies of the impulse response at delays 1, 3, and 5:

$$
\begin{aligned}
y[n] &= h[n-1] + 2h[n-3] + 3h[n-5] && (33)\\
&= \left(\frac{1}{2}\right)^{n-1} u[n-1] + 2\left(\frac{1}{2}\right)^{n-3} u[n-3] + 2\left(\frac{1}{2}\right)^{n-5} u[n-5] && (34)\\
&= \begin{cases} 0 & n < 1 \\ 1 & n = 1 \\ \frac{1}{2} & n = 2 \\ \frac{1}{4} + 2 & n = 3 \\ \frac{1}{8} + 1 & n = 4 \\ \frac{1}{16} + \frac{1}{2} + 2 & n = 5 \\ \frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \frac{1}{2^{n-4}} & n > 5 \end{cases} && (35)\\
&= \begin{cases} 0 & n < 1 \\ 1 & n = 1 \\ \frac{1}{2} & n = 2 \\ \frac{9}{4} & n = 3 \\ \frac{9}{8} & n = 4 \\ \frac{41}{16} & n = 5 \\ \frac{13}{2^{n-4}} & n > 5 \end{cases} && (36)
\end{aligned}
$$

(b) This one is the convolution of two decaying exponentials, but not in exactly the simple form we saw before. First let's write $x[n]$ as a delayed version of a scaled exponential. If we define

$$
\begin{aligned}
z[n] &= \times \left(\frac{1}{3}\right)^n u[n] && (37)\\
x[n] &= \frac{1}{27} z[n-3] && (38)
\end{aligned}
$$

So $(h * x)[n] = \frac{1}{27}(h * z)[n-3]$. Now we've reduced our task to solving $h * z$, for which we have a formula. Since $\frac{1}{3} < \frac{1}{2}$,

$$
\begin{aligned}
(h * z)[n] &= \left(\frac{6}{2^{n+1}} - \frac{6}{3^{n+1}}\right) u[n] && (39)\\
&= \left(\frac{3}{2^n} - \frac{2}{3^n}\right) u[n] && (40)
\end{aligned}
$$

Delaying by 3 and scaling by $\frac{1}{27}$:

$$
\begin{aligned}
(h * x)[n] &= \frac{1}{3^3}\left(\frac{3}{2^{n-3}} - \frac{2}{3^{n-3}}\right) u[n-3] && (41)\\
&= \left(\frac{1/9}{2^{n-3}} - \frac{2}{3^n}\right) u[n-3]. && (42)
\end{aligned}
$$