ECE 345: Linear Systems and Signals — Fall 2020
Professors Sarwate and El Rouayheb — Due: 5 PM, September 11, 2020

Solutions for Homework 1

Problem 1 (Zero-crossings for CT signals)**.** Find the points where these signals take on value 0, i.e. when the cross the time axis. Note that for some of them there may be multiple places where they cross the axis.

(a) $x(t) = 5t - 4$

(b) $x(t) = 2t^2 - 7t + 6$

(c) $x(t) = 3\cos(100\pi t + 50\pi)$

(d) $x(t) = \frac{t-1}{t+1}$

(e) $x(t) = \frac{\sin(4\pi t)}{\pi t}$

You can verify your answers using MATLAB.

Solution:

(a) This is a linear function so it crosses the axis at one point. Setting $x(t) = 0$ and solving we get $t = \frac{4}{5}$.

(b) Use the quadratic formula: $t = \frac{7 \pm \sqrt{7^2 - 4 \cdot 2 \cdot 6}}{4} = \frac{7 \pm 1}{4}$ so $t = 2$ and $t = \frac{3}{2}$.

(c) This is a sinusoid so it crosses the axis an infinite number of times whenever $100\pi t + 50\pi = K\frac{\pi}{2}$ where K is odd. Solving, we get: $t = \frac{K-100}{200}$ for K odd.

(d) When $t = 1$ this is 0. For all other values of t is it strictly positive or strictly negative, except at $t = -1$.

(e) This is the sinc function we saw before, which is 0 when $t = \frac{K}{4}$ for any integer $K \neq 0$

You can verify your answers using MATLAB.

Problem 2 (Limits for CT signals)**.** Find the limiting values as $t \to \infty$ of the following signals.

(a) $x(t) = \frac{1}{t}\log(t)$

(b) $x(t) = t^2 e^{-t}$

(c) $x(t) = e^{-2t}\sin(10\pi t)$

(d) $x(t) = \frac{3t+1}{2t+5}$

You can verify your answers using MATLAB.

Solution:

(a) Since $\log(t)$ grows so much smaller than t, the limit is $\lim_{t\to\infty} x(t) = 0$.

(b) Since e^{-t} decays exponentially fast, it dominates any polynomial so $\lim_{t\to\infty} x(t) = 0$.

(c) Since $\sin(\cdot)$ is bounded and e^{-2t} is decaying, $\lim_{t\to\infty} x(t) = 0$.

(d) Since the constant terms are negligible as $t \to \infty$, $\lim_{t\to\infty} x(t) = \frac{3}{2}$.

Problem 3 (Complex signals)**.** For each of the following complex signals, calculate the magnitude and phase functions analytically.

(a) $x(t) = e^{-j(t/2+3)}$

(b) $x(t) = e^{-jt/2} + 2e^{-jt/3}$

(c) $x[n] = 3e^{j(\pi/6)n}$

(d) $x[n] = 3e^{j(\pi/6)n} + 2e^{j(\pi/9)n}$

Solution:

(a) We have the following:

$$\begin{aligned} \mathfrak{Re}\{x(t)\} &= \cos(t/2+3) && (1)\\ \mathfrak{Im}\{x(t)\} &= \sin(t/2+3) && (2)\\ |x(t)| &= 1 && (3)\\ \measuredangle x(t) &= -t/2-3 && (4) \end{aligned}$$

(b) We have the following:

$$\begin{aligned} \mathfrak{Re}\{x(t)\} &= \cos(t/2) + 2\cos(t/3) && (5)\\ \mathfrak{Im}\{x(t)\} &= -\sin(t/2) - 2\sin(t/3) && (6)\\ |x(t)| &= \left((\cos(t/2)+2\cos(t/3))^2 + (\sin(t/2)+2\sin(t/3))^2\right)^{1/2} && (7)\\ \measuredangle x(t) &= \tan^{-1}\frac{-\sin(t/2)-2\sin(t/3)}{\cos(t/2)+2\cos(t/3)} && (8) \end{aligned}$$

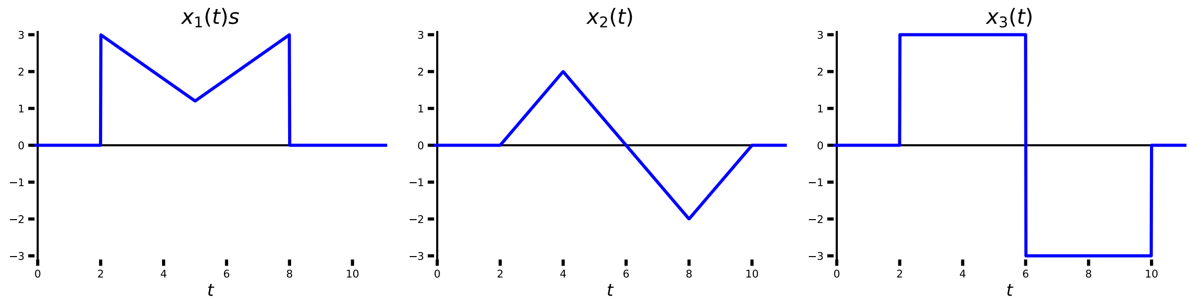
(c) We have the following:

$$\begin{aligned} \mathfrak{Re}\{x[n]\} &= 3\cos((\pi/6)n) && (9)\\ \mathfrak{Im}\{x[n]\} &= 3\sin((\pi/6)n) && (10)\\ |x[n]| &= 3 && (11)\\ \measuredangle x[n] &= (\pi/6)n && (12) \end{aligned}$$

(d) We have the following:

$$\begin{aligned} \mathfrak{Re}\{x[n]\} &= 3\cos((\pi/6)n) + 2\cos((\pi/9)n) && (13)\\ \mathfrak{Im}\{x[n]\} &= 3\sin((\pi/6)n) + 2\sin((\pi/9)n) && (14)\\ |x[n]| &= (3\cos((\pi/6)n) + 2\cos((\pi/9)n))^2 + (3\sin((\pi/6)n) + 2\sin((\pi/9)n))^2 && (15)\\ \measuredangle x[n] &= \tan^{-1}\frac{3\sin((\pi/6)n) + 2\sin((\pi/9)n)}{3\cos((\pi/6)n) + 2\cos((\pi/9)n)}. && (16) \end{aligned}$$

Problem 4 (Steps and ramps, SSTA 1.27)**.** Find formulas for these signals in terms of steps and ramps.



Solution:

A delayed step drops the waveform. A delayed ramp reduces its slope by one. The way to think about these problems is to piece together the waveform. We will describe the process for the first signal; try to justify the other ones on your own:

(a) The signal starts at time 2 with a jump of height 3, so to get there we start with

$$3u(t-2) \tag{17}$$

Now the signal starts going down linearly with a slope of $-2/3$, so we need to subtract off a ramp with coefficient 1:

$$3u(t-2) - \frac{2}{3}r(t-2) \tag{18}$$

When the signal gets to time 5 the value is 0 but then it starts rising again with slope 1, so we need to add $r(t-5)$ to cancel the negative ramp and then another $r(t-5)$ to make it start rising again:

$$x_1(t) = 3u(t-2) - \frac{2}{3}r(t-2) + \frac{4}{3}r(t-5) \tag{19}$$

Then at time 8 we have to cancel the increasing ramp and pull the signal value down from 3 to 0:

$$x_1(t) = 3u(t-2) - \frac{2}{3}r(t-2) + \frac{4}{3}r(t-5) - \frac{2}{3}r(t-8) - 3u(t-8). \tag{20}$$

(b) This signal is $x_2(t) = r(t-2) - 2r(t-4) + 2r(t-8) - r(t-10)$

(c) This signal is $x_3(t) = 3u(t-2) - 6u(t-6) + 3u(t-10)$

Problem 5 (CT periodic signals, SSTA 1.26)**.** Determine the period of each of the following waveforms. If the signal is not periodic, explain why.

(a) $x(t) = \sin(2t)$

(b) $x(t) = \cos\left(\frac{\pi}{3}t\right)$

(c) $x(t) = \cos^2\left(\frac{\pi}{3}t\right)$

(d) $x(t) = \cos(4\pi t + \pi/3) - \sin(4\pi t + \pi/3)$

(e) $x(t) = \cos\left(\frac{4}{\pi}t + \frac{\pi}{6}\right) - \sin\left(4\pi t + \frac{\pi}{6}\right)$

Solution:

The useful fact to keep in mind is that the period of $A\cos(\omega t + \theta)$ is $\frac{2\pi}{\omega}$ as long as $\omega \neq 0$. Applying this fact, we get the answers:

(a) The period is $\frac{2\pi}{2} = \pi$.

(b) The period is $\frac{2\pi}{\pi/3} = 6$.

(c) Using a trig identity again: $\cos^2\left(\frac{\pi}{3}t\right) = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi}{3}\right)$, so the period is $\frac{2\pi}{2\pi/3} = 3$.

(d) This one looks a little trickier, but it's not too bad. We need to use another trig identity: $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$, where $A = 4\pi t + 60°$ and $B = 45°$. Thus

$$\cos(4\pi t + 60°) - \sin(4\pi t + 60°) = \sqrt{2}\cos(4\pi t + 105°) \tag{21}$$

and the period is $\frac{2\pi}{4\pi} = \frac{1}{2}$.

(e) Sometimes the sum of periodic functions is not periodic. Here, the first term $\cos((4/\pi)t + \pi/6)$ has period $\frac{2\pi}{4/\pi} = \frac{\pi^2}{2}$ and the second term $\sin(4\pi t + \pi/6)$ has period $\frac{2\pi}{4\pi} = \frac{1}{2}$. Since one of the periods is not an integer multiple of the other, the sum of these two sinusoids is not periodic.

Problem 6 (DT periodic signals, SSTA 7.3)**.** Compute the fundamental periods and fundamental angular frequencies of the following signals. If the signal is not periodic, explain why.

(a) $7\cos(0.16\pi n + 2)$

(b) $7\cos(0.16n + 2)$

(c) $3\cos(0.16\pi n + 1) + 4\cos(0.15\pi n + 2)$

Solution:

For a discrete-time sinusoid $A\cos(\Omega n + \theta)$, the fundamental period is

$$N_0 = \frac{2\pi k}{\Omega}, \tag{22}$$

where k is the smallest integer value that results in an integer value for N_0. If no integer value of k exists that satisfies this condition, then the sinusoid is not periodic.

(a) We can calculate:

$$N_0 = \frac{2\pi k}{\Omega} = \frac{2\pi k}{0.16\pi} = \frac{25}{2}k. \tag{23}$$

Now to figure out if an integer k will make N_0 an integer, we can see $k = 1$ won't cut it but $k = 2$ will, so $N_0 = \frac{25}{2} \times 2 = 25$ samples.

To get the fundamental angular frequency, we just take 2π and divide by N_0:

$$\Omega_0 = \frac{2\pi}{N_0} = \frac{2\pi}{25} \text{ radians/sample.} \tag{24}$$

Units are important! Frequency is measured in radians (not degrees) and for DT signals we don't have a scale for the time axis, so it's radians per sample.

(b) We can go through the same process:

$$N_0 = \frac{2\pi k}{\Omega} = \frac{2\pi k}{0.16} = \frac{25\pi}{2}k. \tag{25}$$

Because π is irrational, there is no integer value for k that will make N_0 an integer. So this signal is *aperiodic* (not periodic).

(c) We can calculate the period for the first and second terms.

$$N_{0,1} = \frac{2\pi k}{0.16\pi} = \frac{25}{2}k \Longrightarrow N_{0,1} = 25 \text{ samples} \tag{26}$$
$$N_{0,2} = \frac{2\pi k}{0.15\pi} = \frac{40}{3}k \Longrightarrow N_{0,2} = 40 \text{ samples} \tag{27}$$

So we have the sum of signals with period 25 and 40. To get the period of the sum, we need to look at the *least common multiple* of 25 and 40, which is 200 samples. So the first term goes through 8 cycles in 200 samples and the second term goes through 5 cycles in 200 samples.

Now to get the fundamental angular frequency we have

$$\Omega_0 = \frac{2\pi}{N_0} = \frac{2\pi}{200} = 0.01\pi \text{ radians/sample} \tag{28}$$

Problem 7 (CT power and energy)**.** Determine power and energy for each of the following CT signals to decide if they are energy-type, power-type, or neither.

(a) $x_1(t) = e^{-2t}u(t)$

(b) $x_2(t) = e^{j(2t+\pi/4)}$

(c) $x_3(t) = \cos(t)$

Solution:

Recall that $|x(t)|^2$ is the *instantaneous power* for a CT signals. Energy signals have

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty \tag{29}$$

and power signals have

$$0 < \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt < \infty \tag{30}$$

(a) For our first signal, the $u(t)$ means we only have to integrate over $[0, \infty)$. On this interval e^{-2t} is decaying (exponentially fast) so the area under the curve of $|x_1(t)|^2 = e^{-4t}u(t)$ should be finite. So a good guess would be that it's energy-type, but we have to verify:

$$\mathcal{E}_1 = \int_{-\infty}^{\infty} (e^{-2t})^2 u(t) dt = \int_0^{\infty} e^{-4t} dt = \left[-\frac{1}{4} e^{-4t} \right]_0^{\infty} = \frac{1}{4}. \tag{31}$$

Since it's energy-type, $\mathcal{P}_1 = 0$.

(b) This is a complex exponential, so to find the instantaneous power we can use magnitude-phase form or write it out into real and imaginary parts:

$$|x_2(t)|^2 = x_2(t)x_2^*(t) = e^{j(2t+\pi/4)}e^{-j(2t+\pi/4)} = 1 \tag{32}$$
$$|x_2(t)|^2 = |\cos(2t + \pi/4) + j\sin(2t + \pi/4)| = \cos^2(2t + \pi/4) + \sin^2(2t + \pi/4) = 1. \tag{33}$$

You can do either, but the first one is a bit simpler. Since the instantaneous power is 1 at all times t, the average power is also 1, so $\mathcal{P}_2 = 1$. This is a power-type signal so $\mathcal{E} = \infty$.

(c) Since $\cos(t)$ is periodic (with period 2π), we know it is a power-type signal, so $\mathcal{E}_3 = \infty$. The instantaneous power is $|\cos(t)|^2 = \cos^2(t)$. Integrating over a single period:

$$\mathcal{P}_3 = \frac{1}{2\pi} \int_0^{2\pi} \cos^2(t) dt \tag{34}$$

Now you have to dredge up more trig identities from your memory to use a double angle formula:

$$\cos(2t) = \cos^2(t) - \sin^2(t) = 2\cos^2(t) - 1 \tag{35}$$

$$\cos^2(t) = \frac{1}{2}\left(1 + \cos(2t)\right). \tag{36}$$

Now,

$$\mathcal{P}_3 = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2}\left(1 + \cos(2t)\right) dt \tag{37}$$

$$= \frac{1}{2} + \frac{1}{4\pi} \int_0^{2\pi} \cos(2t) dt \tag{38}$$

$$= \frac{1}{2} \tag{39}$$

Note that the integral is just integrating $\cos(2t)$ over two periods so the area under the curve is 0.

This might seem like a lot of work but there are actually several short cuts you could use to get the answer. This approach highlights the importance of double angle formulas, which will be useful elsewhere. This trick of converting $\cos^2(t)$ into $\frac{1}{2}(1 + \cos(2t))$ appears quite often so it's good to stash it away in your memory for future use.

Problem 8 (DT signals)**.** For each of the following DT signals determine (analytically) the power and energy to decide if they are energy-type, power-type, or neither.

(a) $x_1[n] = \left(\frac{1}{2}\right)^n u[n]$

(b) $x_2[n] = e^{j(\pi/2n + \pi/8)}$

(c) $x_3[n] = \cos\left(\frac{\pi}{4}n\right)$

Solution:

Recall that $|x[n]|^2$ is the instantaneous power for a DT signals. Energy signals have

$$\sum_{-\infty}^{\infty} |x[n]|^2 < \infty \tag{40}$$

and power signals have

$$0 < \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} |x[n]|^2 < \infty \tag{41}$$

(a) Our first example is the DT analogue of the decaying exponential: $\left(\frac{1}{2}\right)^n u[n]$ is decaying exponentially fast over time. Again, this suggests it is probably an energy-type signal. We can test this:

$$\mathcal{E}_1 = \sum_{n=-\infty}^{\infty} \left| \left(\frac{1}{2}\right)^n u[n] \right|^2 \tag{42}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n} \tag{43}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{n} \tag{44}$$

$$= \frac{1}{1 - 1/4} \tag{45}$$

$$= \frac{4}{3} \tag{46}$$

So $\mathcal{P}_1 = 0$.

This solution indicates another important mathematical tool that we will use over and over again in this course: the sum of the geometric series:

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}. \tag{47}$$

Because in DT signals we will end up doing a ton of sums like this, we will encounter the geometric series in many forms to get clean formulas for various things.

(b) Again, we can look at the instantaneous power:

$$\left| e^{j(\pi/2n + \pi/8)} \right|^2 = 1 \qquad \forall n \tag{48}$$

(where $\forall$ stands for "for all"). Since the instantaneous power is 1 at all times, the average power $\mathcal{P}_2 = 1$ and $\mathcal{E}_2 = \infty$.

(c) First we should check if this signal is periodic: $N_0 = \frac{2\pi k}{\pi/4} = 8k$ so it is periodic with period 8. Therefore it is power-type and $\mathcal{E}_3 = \infty$. We can just hand-calculate the power:

$$\mathcal{P}_3 = \frac{1}{8}\big(\cos^2(0) + \cos^2(\pi/4) + \cos^2(\pi/2) + \cos^2(3\pi/4) + \cos^2(\pi) + \cos^2(5\pi/4) + \cos^2(3\pi/2) + \cos^2(7\pi/4)\big) \tag{49}$$

$$= \frac{1}{8}\left(2 + 4\left(\frac{1}{\sqrt{2}}\right)^2 \right) \tag{50}$$

$$= \frac{1}{2}. \tag{51}$$