Signals:

ECE 345 Fall 2017
Rutgers University
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Objectives:

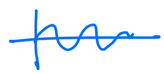
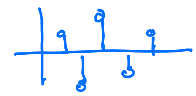
1) Be able to classify signals by their properties: discrete/continuous time, periodic/aperiodic, even/odd, etc.
2) Be able to calculate properties of signals such as their period, power, or energy.
   - understand the difference between power and energy
3) Manipulate signals through simple operations like delay, time dilation, and reversal.
4) Understand and use basic signal constructions such as the unit step function, complex exponentials and impulse functions*

* a more detailed treatment of impulse functions will come later.

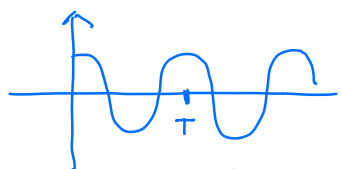
Types of signals

There are many types of signals that we will encounter in this class:

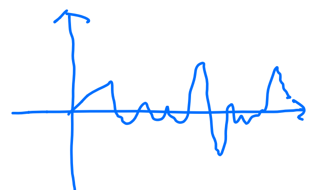
- discrete time and continuous time

  $x(t)$  $x[n]$ 

- periodic and aperiodic

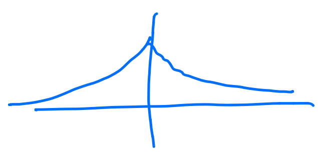


periodic
repeats every T

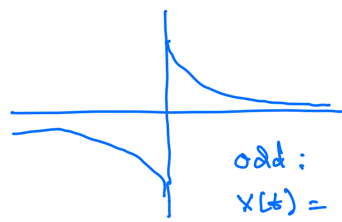


aperiodic

- even and odd



even: $x(t) = x(-t)$



odd: $x(t) = -x(-t)$

We also want to be able to <u>measure</u> different properties of signals

- energy / power

$$\text{energy} = \int_{-\infty}^{\infty} x(t)^2 \, dt$$

$$\text{power} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)^2 \, dt$$

- period

$$x(t) = x(t+T) \quad \text{for some } T$$

We want to be able to understand how simple signal manipulations work:

- delay $x(t) \longrightarrow x(t-2)$
- compressing ("squishing") the time axis: $x(t) \longrightarrow x(2t)$
- expanding ("stretching") the time axis: $x(t) \longrightarrow x(t/2)$

Periodic and aperiodic signals:

As we heard in the introduction to the course, sin / cos waves are going to be a big part of this course. More generally, we will see that we will have to treat signals that are periodic differently from signals that are not periodic (also called aperiodic).

Def [**Def:** A continuous time signal $x(t)$ is periodic with period T if

$$x(t) = x(t+T)$$

for all t. A discrete-time signal $x[n]$ is periodic with period N if

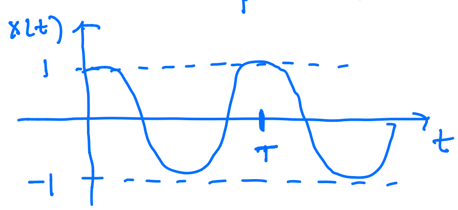
$$x[n] = x[n+N]$$

!!! [**Note:** If $x(t)$ is periodic with period T it is also periodic with period $2T$, $3T$, etc. Make sure you understand why!

Def [**Def.** The period of a periodic signal $x(t)$ (or $x[n]$) is the smallest T (or N) for which the $x(t)$ (or $x[n]$) is periodic with period T (or N).

Ex [**Example:** $x(t) = \cos(2\pi f t)$ f in Hertz $= \frac{1}{\text{sec.}}$

what is the period of $x(t)$?



Graphical approach:

1) plot the function
2) eyeball it
3) find the point where it starts repeating

T is the first point >0 where $x(t) = 1$

$$\cos(2\pi f T) = 1$$
$$2\pi f T = 2\pi$$
$$T = 1/f \Rightarrow \text{period is } 1/f \text{ seconds}$$

Analytical approach: use the definition & do algebra:

$$x(t) = x(t+T)$$

$$\cos(2\pi f t) = \cos(2\pi f(t+T))$$

☆ $\cos(\cdot)$ has period 2π:

$$\cos(2\pi f t) = \cos(2\pi f t + 2\pi f T)$$

$$\Rightarrow 2\pi f T = 2\pi$$

$$\Rightarrow T = 1/f$$

!!! **Protip:** keep track of units like Hertz, seconds, Watts, Joules, etc. It can help debug your answer. If the units aren't right, you made an error somewhere!

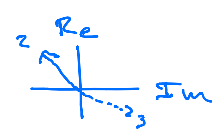
Ex **Example:** $x[n] = e^{j\omega n}$. For what values of ω is this signal periodic?

!!! ☆ This is a complex-valued signal.

First we have to understand the question: why might $x[n]$ not be periodic for some values of ω?

Easiest approach: try some examples. Take $\omega = 2$.

$$x[0] = 0$$
$$x[1] = e^{j2}$$
$$x[2] = e^{j4}$$
$$x[3] = e^{j6} \ldots$$



doesn't look like it will repeat.

!!! Note: this is not a proof, but doing some examples can build intuition.

What does the example suggest? We need to get these complex numbers to repeat. Thinking of the magnitude-phase representation we need to get the phases/angles to repeat:

$$\omega, \quad 2\omega, \quad 3\omega, \quad 4\omega, \quad \ldots$$

But we actually want them to repeat "mod 2π": every time the phase goes around a full circle it goes back to 0. Therefore

$$\omega \pmod{2\pi}, \quad 2\omega \pmod{2\pi}, \quad \ldots$$

should be periodic

From the definition:

$$\omega n \pmod{2\pi} = \big(\omega(n+N)\big) \pmod{2\pi}$$

$$= (\omega n + \omega N) \pmod{2\pi}$$

or $\quad \omega N = m\, 2\pi \quad$ for some m.

So $\quad \omega = \dfrac{2\pi m}{N} \quad$ for integers m and n

Putting it together:

$$x[n] = e^{j\omega n}$$

is periodic with period N if there is an integer m such that

$$\omega = \frac{2\pi m}{N} \quad \text{radians}$$

units!

Not **Notation / jargon:** To make things more succinct, we will be introducing a bit of jargon and acronyms. It's good to get used to it sooner rather than later:

CT = continuous time

DT = discrete time

Also we will be using a bit of math notation:

$\mathbb{R}$ = the set of real numbers

$\mathbb{C}$ = the set of complex numbers

$\mathbb{Z}$ = the set of integers

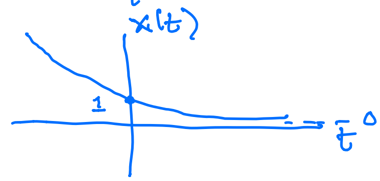
$\in$ = "is in" or "is an element of"

$x \in \mathbb{R}$ = "x is a real number"

$n \in \mathbb{Z}$ = "n is an integer"

Ex **Example:** $x(t) = e^{-t}$. What is the period of $x(t)$?

Trick question! Draw a picture:



This is pretty clearly not periodic.

Try **Check yourself:** Make up some functions and see if they are periodic or not. If they are, find their period.

Main

Even and odd signals:

Symmetry is another important property.

Def

Def: A CT signal $x(t)$ is <u>even</u> if

$$x(-t) = x(t)$$

and is <u>odd</u> if

$$x(-t) = -x(t)$$

A DT signal $x[n]$ is <u>even</u> if

$$x[-n] = x[n]$$

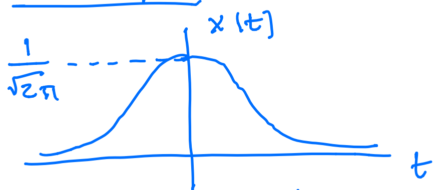
and is <u>odd</u> if

$$x[-n] = -x[n]$$

As always, the key to understanding is through drawing pictures.

Ex

Example:



Where have you seen this before?

$$x(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}$$

Looks symmetric, probably even. Let's check:

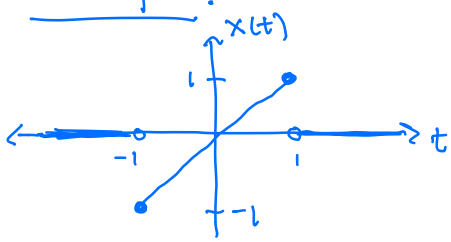
$$x(-t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-t)^2}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}$$

$$= x(t)$$

Checks out.

Example:



$$x(t) = \begin{cases} t & -1 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Looks odd...

$$x(-t) = \begin{cases} -t & -1 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= -x(t)$$

Main "Most" functions are neither even nor odd. So why do we care so much?

<u>Fact</u>: Any signal can be written as the sum of an even signal and an odd signal:

$$x(t) = x_{even}(t) + x_{odd}(t)$$
$$x[n] = x_{even}[n] + x_{odd}[n]$$

Not <u>Notation</u>: The book uses this notation so we'll stick with it.

$$\mathcal{E}v\{x(t)\} \quad \text{even part of } x(t)$$
$$\mathcal{O}d\{x(t)\} \quad \text{odd part of } x(t)$$

Main If you are seeing this fact for the first time it might seem pretty <u>odd</u>, but <u>even</u>tually it will start to make sense. The trick is that we can construct the even and odd parts from $x(t)$ (or $x[n]$)

$$\mathcal{E}v\{x(t)\} = \frac{1}{2}\left(x(t) + x(-t)\right) = x_{even}(t)$$
$$\mathcal{O}d\{x(t)\} = \frac{1}{2}\left(x(t) - x(-t)\right) = x_{odd}(t)$$

So
$$\begin{aligned} x_{even}(t) + x_{odd}(t) &= \tfrac{1}{2}x(t) + \tfrac{1}{2}x(-t) \\ &\quad + \tfrac{1}{2}x(t) - \tfrac{1}{2}x(-t) \\ &= x(t) \end{aligned}$$

and

$$x_{even}(-t) = \frac{1}{2}\left(x(-t) + x(t)\right) = x_{even}(t)$$

$$x_{odd}(-t) = \frac{1}{2}\left(x(-t) - x(t)\right) = -x_{odd}(t)$$

This is a pretty neat trick, and what it will let us do is analyze the even and odd parts separately.

The same trick works for DT signals:

$$\mathcal{E}v\{x[n]\} = \frac{1}{2}\left(x[n] + x[-n]\right)$$
$$\mathcal{O}d\{x[n]\} = \frac{1}{2}\left(x[n] - x[-n]\right)$$

Try

Check yourself: make up your own examples of even functions, odd functions, and functions that are neither even nor odd. For the last group, compute their even and odd parts.

!!!

Protip: It's good to have examples in your mind for different signal types or for different signal properties. That way when you encounter a new signal you have an example to compare it with.

Main

Energy and power

We will often talk about the energy or power in a signal. After spending so much time arguing that signals are everywhere and that lots of measurements can be thought of as signals, it should be pretty clear that Joules (energy) and Watts (power) are not concepts that apply to all signals.

Some reasons we talk about energy and power are

1) voltage signals are pretty important, as are EM waves. If $V(t)$ is a voltage signal then it dissapates

$$\frac{V(t)^2}{R}$$

watts (joules/sec) over a resistor of R ohms.

2) The formulas still measure important properties of more general signals. In those settings we won't use Joules and watts as units, but instead we will just think of "energy" as having some abstract unit and power as energy/time

Phil **Philosophical note:** we are doing the same thing with time. Remember that some signals are measured over space, not time.

But for this class we think about a generic signal as a function over time. If it helps, you can think of a generic signal as a voltage signal over a unit resistor.

Def **Def.** The instantaneous power signal for signals is the squared magnitude of the signal:

Real CT signal $x(t) \longrightarrow |x(t)|^2 = x(t)^2$

Complex CT signal $x(t) \longrightarrow |x(t)|^2 = x(t)\, x^*(t)$

Real DT signal $x[n] \longrightarrow |x[n]|^2 = x[n]^2$

Complex DT signal $x[n] \longrightarrow |x[n]|^2 = x[n]\, x^*[n]$

where $x^*(t)$ and $x^*[n]$ are complex conjugates.

Main The instantaneous power signal is also a signal over time. Power is $\text{energy}/\text{time}$, so to get the energy of a signal, we just integrate/sum over time.

Def The total energy of a CT signal $x(t)$ is

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

The total energy of a DT signal $x[n]$ is

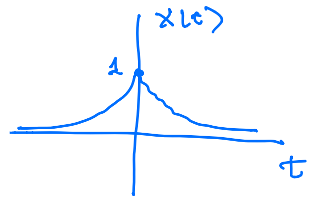
$$\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Main The total energy in a signal can be finite or infinite. A few examples should help clarify things.

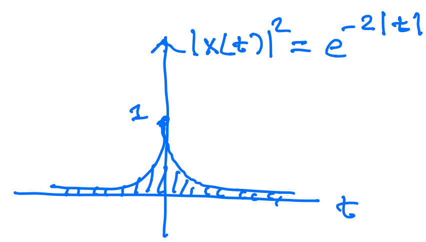
Ex **Example: real CT signal energy.**

Suppose $x(t) = e^{-|t|}$. What is its energy $\mathcal{E}_x$?

Draw a picture:



signal



inst. power

So we need to do the following integral:

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= \int_{-\infty}^{\infty} e^{-2|t|} \, dt$$

$$= 2\int_0^{\infty} e^{-2t} \, dt$$ by symmetry: a useful trick!

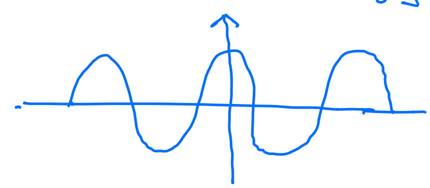
$$= 2\left[-\frac{1}{2} e^{-2t}\right]_0^{\infty}$$

$$= 2\left(0 - \left(-\frac{1}{2}\right)\right)$$

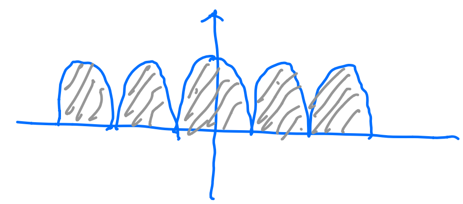
$$= 1.$$

Example: CT signal with infinite energy

$x(t) = \cos(\omega t)$



$|x(t)|^2 = \cos^2(\omega t)$



Area under the curve is $\longrightarrow \infty$

So

$$\mathcal{E}_x = \infty$$

Example: Complex CT signal

$$x(t) = e^{-|t|} \cos(\omega t) + j e^{-|t|} \sin(\omega t)$$

We need $x^*(t)$ to get $|x(t)|^2$... or do we?

$$x(t) = \underset{\substack{\uparrow \\ \text{mag.}}}{e^{-|t|}} \cdot \underset{\substack{\uparrow \\ \text{phase}}}{e^{j\omega t}}$$

$$|x(t)|^2 = e^{-2|t|}$$

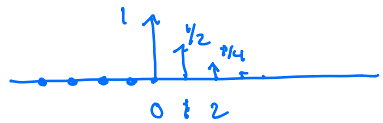
$\mathcal{E}_x = 1$ (from the earlier example)

Example: real DT signal

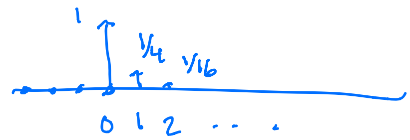
Suppose $x[n] = \begin{cases} \frac{1}{2^n} & n \geq 0 \\ 0 & n < 0 \end{cases}$

Then $|x[n]|^2 = \begin{cases} \frac{1}{2^{2n}} & n \geq 0 \\ 0 & n < 0 \end{cases}$

Drawing a picture...



$x[n]$



$|x[n]|^2$

Now algebra:

$$\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= \sum_{n=0}^{\infty} |x[n]|^2 \qquad \text{since } |x[n]|^2 \text{ is } 0 \text{ for } n \leq 0$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \qquad \text{substituting in}$$

$$= \sum_{n=0}^{\infty} \frac{1}{4^n} \qquad \text{since } 2^{2n} = (2^2)^n$$

$$= \frac{1}{1-1/4} \qquad \text{this is a geometric series}$$

$$= \frac{4}{3}$$

!!! ⌈ We are going to see sums like this a lot so brush up on your geometric series manipulations.

Main From one example it seemed that periodic signals have infinite energy. We can show why this is true in general.

Suppose $x(t)$ is a signal with period T. Then

$$|x(t)|^2 = |x(t+T)|^2$$

so the instantaneous power is also periodic with period T. Then

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \lim_{N\to\infty} \int_{-NT}^{NT} |x(t)|^2 dt$$

we're looking at the integral over $[-NT, NT]$ and then letting N grow

$$= \lim_{N\to\infty} 2N \cdot \int_{-T}^{T} |x(t)|^2 dt$$

since $x(t)$ repeats $2N$ times in $[-NT, NT]$

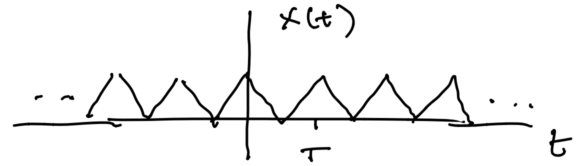
$$= \lim_{N\to\infty} 2N \cdot \alpha$$

where $\alpha = \int_{-T}^{T} |x(t)|^2 dt \geq 0$ since $|x(t)|^2 \geq 0$

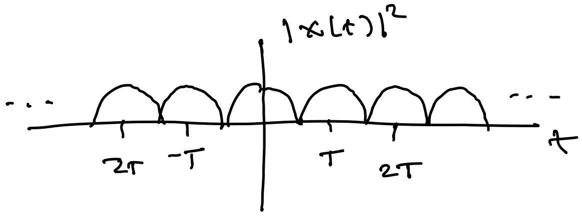
$$= \infty$$

unless $\alpha = 0$ but that would mean $x(t) = 0$

Confused? A picture helps!



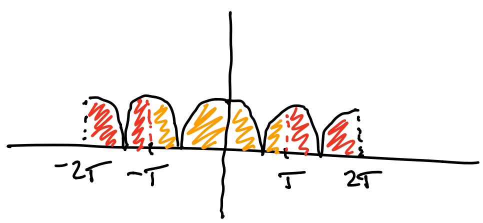
Here is our periodic function



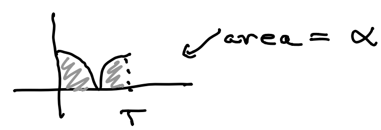
Here is the instantaneous power.

It's also periodic with period T

Suppose $N=2$. Then we look at the signal from $[-2T, 2T]$



The area under the curve is $4\times$ the area under one period:



So as we let N get larger we will get $2N\alpha$ area from $2N$ repeats.

!!! All periodic signals have infinite energy but not all infinite energy signals are periodic. Example: $x(t) = t$.

Main All of this suggests that energy is not the "right way" to measure periodic signals. From the argument above, we see that for periodic signals the energy grows linearly in the number of copies as we extend the integral to cover more of the time axis. That is, the energy per unit time is constant for periodic signals. This motivates our definition of power.

Def. **Def.** The time-averaged power of a signal $x(t)$ or $x[n]$ is

$$[CT] \quad P_x = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$[DT] \quad P_x = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 dt$$

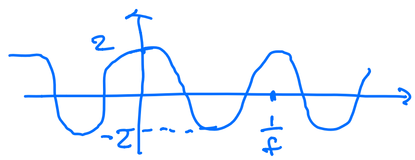
Def. A signal $x(t)$ or $x[n]$ is called energy-type or a finite energy signal if $\mathcal{E}_x < \infty$. It is called power-type if $P_x < \infty$.

!!! Signals can be energy-type and/or power-type. They may also be neither energy- nor power-type. Example: $x(t) = t$.

Ex

Example: $x(t) = 2\cos(2\pi f t)$. What is the power of $x(t)$?

Drawing a picture...



Period is $T = 1/f$.

$$P_x = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} 4\cos^2(2\pi f t)\,dt$$

$$\int_{-T}^{T} 4\cos^2(2\pi f t)\,dt$$

$$= 4 \cdot 2\int_0^T \cos^2(2\pi f t)\,dt$$

by symmetry

$$= 4\int_0^T (1 + \cos(4\pi f t))\,dt$$

By $\cos(2A) = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$

$\cos^2 A = \frac{1}{2}(\cos(2A) + 1)$

$$= 4T + 4\Big[\sin(4\pi f t)\Big]_0^T$$

$$= 4T + 4\sin(4\pi f T)$$

$$= 4T + 4\sin(4\pi\, f/f)$$

since $T = 1/f$

$$= 4T$$

since $\sin(4\pi) = 0$

$$P_x = \lim_{T\to\infty} \frac{1}{2T} \cdot 4T$$

$$= 2$$

Example: $x[n] = 3e^{-j\omega n}$

Then $|x[n]|^2 = 9$ since $|e^{-j\omega n}| = 1$

This is a constant, so

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} (2N+1) \cdot 9$$

$$= 9$$

Example: $x[n] = \begin{cases} e^{-n(\ln 3 + j\omega)} & n \geq 0 \\ 0 & n < 0 \end{cases}$

To find the power let's get this into magnitude - phase form first:

$$x[n] = e^{-n\ln 3} \cdot e^{-j\omega n}$$

$$= e^{\ln \frac{1}{3^n}} \cdot e^{-j\omega n} \quad \text{by basic log properties}$$

$$= \frac{1}{3^n} e^{-j\omega n} \quad \text{since } e^{\ln \alpha} = \alpha$$

$$|x[n]|^2 = \frac{1}{9^n}$$

So $$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} \frac{1}{9^n}$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \left(\sum_{n=0}^{\infty} \frac{1}{9^n} - \sum_{n=N+1}^{\infty} \frac{1}{9^n} \right)$$

geometric series trick

$$= \lim_{N\to\infty} \frac{1}{2N+1} \left(\frac{1}{1-1/9} - \frac{1}{9^{N+1}} \sum_{n=0}^{\infty} \frac{1}{9^n} \right)$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \left(\frac{1}{1-1/9} - \frac{1/9^{N+1}}{1-1/9} \right)$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \frac{1-1/9^{N+1}}{1-1/9}$$

$$< \frac{1}{1-1/9} = \frac{9}{8}$$

$$= 0$$

Not: You may be asking why I didn't stop on line 2 above and say

$$\sum_{n=0}^{N} \frac{1}{9^n} < \sum_{n=0}^{\infty} \frac{1}{9^n} = \frac{1}{1-1/9} = \frac{9}{8}$$

and then see immediately that $P_x = 0$.

The reason was to highlight the geometric series trick:

$$\sum_{n=M}^{\infty} \frac{1}{\alpha^n} = \sum_{n=0}^{\infty} \frac{1}{\alpha^{M+n}}$$

$$= \frac{1}{\alpha^M} \sum_{n=0}^{\infty} \frac{1}{\alpha^n}$$

$$= \frac{1/\alpha^M}{1-1/\alpha}$$

We will be using tricks like this often when dealing with DT signals, so it is a good idea to get comfortable with them.

Not Another thing to review from high school are trigonometric identities like $\cos(A \pm B)$, $\sin(A \pm B)$, and their friends. We won't be doing very complicated things but having double angle identities "at your fingertips" will be very helpful.

Try <u>Check yourself</u>: make up some functions and calculate their energy and power. Find examples of energy-type and power-type signals. Find a signal that is not periodic but has $P_x > 0$.

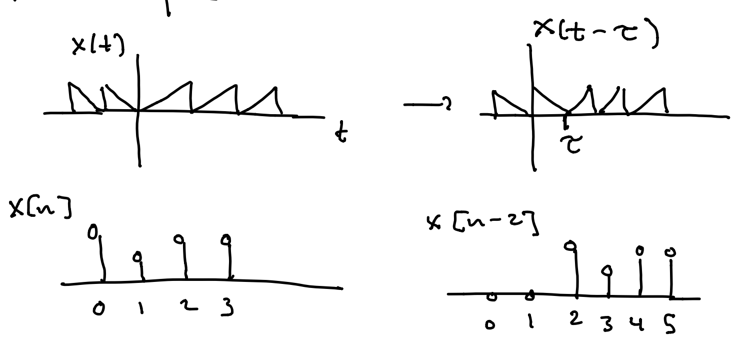
Phil Quantities that we measure about signals are often used as inputs to more advanced processes. For example, energy or power might be used by a machine learning algorithm to help label or classify the signal. For example, energy can help detect whether a person is speaking into a microphone — the energy in the audio signal is higher when someone is speaking. In machine learning these quantities that we compute from the signal are called <u>features</u>.

Main

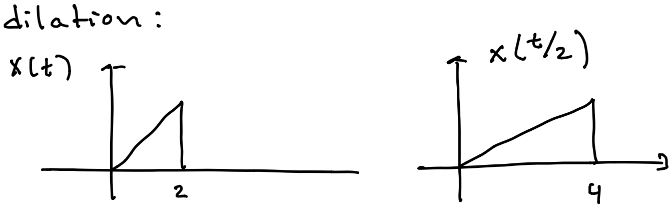
Simple signal manipulations

Systems are things that <u>transform</u> signals. In order to understand systems it helps to think about simple transformations:

delay :

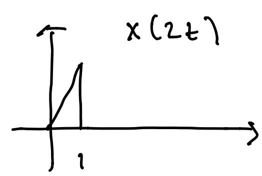


time dilation:

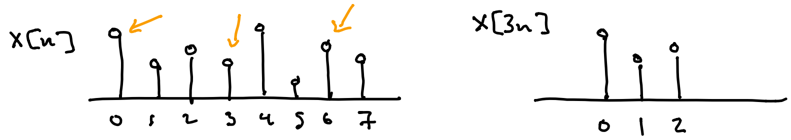


Note: $x(\alpha t)$

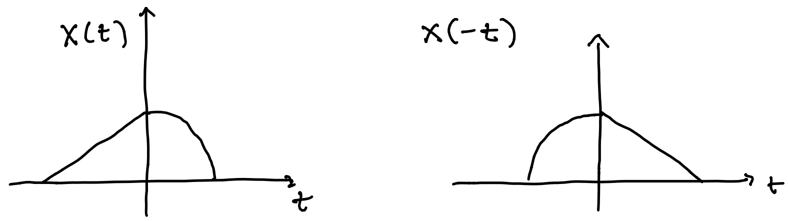
- stretches for $\alpha < 1$
- squishes for $\alpha > 1$

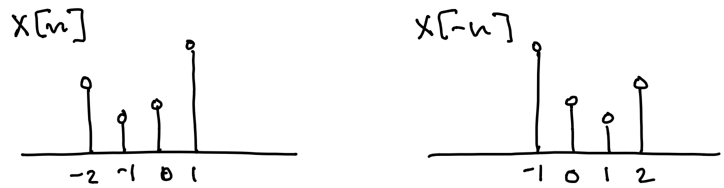


this is a little counterintuitive at first decimation



time reversal

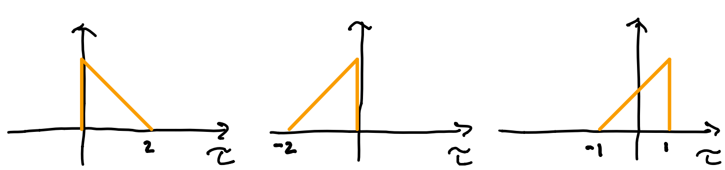




We will be doing a lot of these manipulations in this class. In particular, we will see integrals like

$$\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

this is a "flipped" & shifted version of h:



$h(\tau)$	$h(-\tau)$	$h(-\tau+t)$
original	time-reversed ("flipped")	shifted (for $t=1$)

We will also see sums that look the same:

$$\sum_{m=-\infty}^{\infty} x[m]\, h[n-m]$$

↑ flipped and shifted again

Try

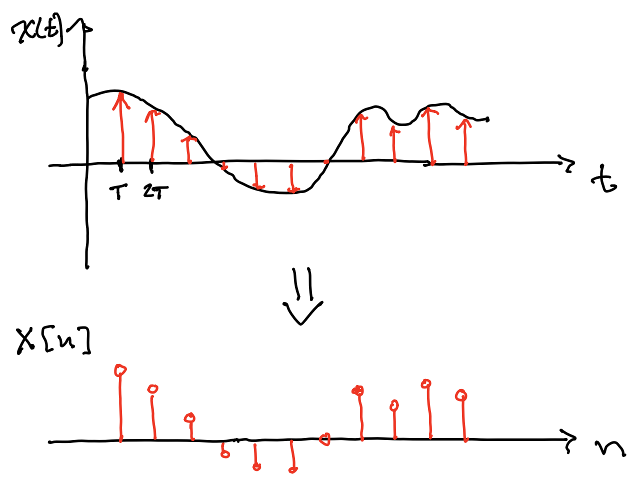
<u>Check yourself</u>: take some functions (maybe the ones from earlier self-checks) and try these signal manipulations. Write the formula and then sketch the function to build an intuition about what these transformations are doing.

Main

Here is another one that we will come to later in the course:

Sampling: given a <u>period</u> T, we make a DT signal from a CT signal:

$$x[n] = x(nT)$$



Question: can we recover $x(t)$ from $x[n]$?

Sometimes... (for later in the course)

Main

Special Signals

There are lots of "special" signals that we will encounter over and over again.

Complex exponentials:

remember what this means?

$$x(t) = C e^{at} \quad \text{where } C, \alpha \in \mathbb{C}$$

We saw an earlier example of periodic complex exponentials:

$$x(t) = e^{j\omega_0 t}$$
$$= \cos(\omega_0 t) + j \sin(\omega_0 t)$$

Note that this Euler-esque version automatically gives us the even and odd parts of the signal:

$$\mathcal{E}v\{e^{j\omega_0 t}\} = \cos(\omega_0 t)$$
$$\mathcal{O}d\{e^{j\omega_0 t}\} = j\sin(\omega_0 t)$$

Not

Defining even/odd for complex signals is a little more complicated – we need to look at "conjugate symmetry." We'll see more later.

Ex

Example: Why is $x(t) = e^{j\omega_0 t}$ periodic? What is its period?

Note that $e^{j(\omega_0 t + 2\pi)} = e^{j\omega_0 t}$

so: $e^{j\omega_0 (t + \frac{2\pi}{\omega_0})} = e^{j\omega_0 t}$

This means $x(t)$ is periodic with period $T_0 = \frac{2\pi}{\omega_0}$.

Try

Come up with an argument to show that T_0 is the period of $x(t)$. That is, show that $x(t)$ is <u>not</u> periodic for any period $T < T_0$.

Main

We can also introduce a phase shift:

$$x(t) = e^{j(\omega_0 t + \varphi)}$$

Sinusoids: sine and cosine functions are also fundamental building blocks:

$$x(t) = \underset{\text{amplitude}}{A} \cos(\underset{\substack{\text{frequency} \\ \text{(in radians)}}}{\omega_0} t + \underset{\text{phase shift}}{\varphi})$$

Not

The frequency of a signal can be given in Hertz or in radians/sec. This is just a matter of units since we can convert one to the other:

$$\underset{\text{omega}}{\omega} = 2\pi f$$

So $20 \text{ Hz} = 40\pi$ rad/sec.

People who work in communications or audio signal processing like Hz because the radio spectrum (and audio frequencies) are

usually given in Hz. More general signal processing is often done in rad/sec with ω denoting frequency.
The important thing to remember is how to convert from one to the other. It takes a little practice...

Main

If we dig a little deeper with Euler's relation we can get a couple of other tricks:

$$e^{j\omega t} + e^{-j\omega t} = 2\cosh(j\omega t)$$

$$= \cos(\omega t) + j\sin(\omega t) + \cos(-\omega t) + j\sin(-\omega t)$$

$$= \cos(\omega t) + j\sin(\omega t) + \cos(\omega t) - j\sin(\omega t)$$

$$= 2\cos(\omega t)$$

This lets us do some factoring tricks
— see Example 1.5 in the book

For general complex exponentials, if

$$C = |c|\, e^{j\theta} \qquad a = r + j\omega_0$$

Then

$$C e^{\alpha t} = |c| e^{rt} e^{j(\omega_0 t + \theta)}$$

$$= |c| e^{rt} \cos(\omega_0 t + \theta) + j|c| e^{rt} \sin(\omega_0 t + \theta)$$

What about discrete-time signals?

Complex exponential: $x[n] = C e^{\beta n}$

but we usually take the form

$$x[n] = C \alpha^n \qquad \alpha = e^{\beta}$$

we also have the special case

$$x[n] = e^{j\omega_0 n}$$

and similarly

$$A\cos(\omega_0 n + \theta) = \frac{A}{2} e^{j\theta} e^{j\omega_0 n} + \frac{A}{2} e^{-j\theta} e^{-j\omega_0 n}$$

Try: Verify the formula. Is this signal energy- or power-type? What is its energy and power?

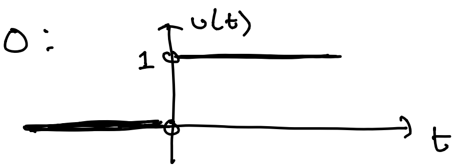
Main: We saw via an earlier example that the complex exponential in DT is periodic when

$$\frac{\omega_0}{2\pi} = \frac{m}{N} \qquad N \text{ is the period}, \quad m \in \mathbb{Z}$$

remember this notation?

The unit step function:

The unit step function "steps" from 0 to 1 at time 0:



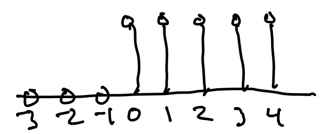
Mathematically:

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

Note: this is discontinuous at 0 and is actually not defined for $t=0$

For discrete time we have a similar thing:

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



Phil

- The "old school" name for $u(t)$ or $u[n]$ is the Heaviside step function, named after Oliver Heaviside (1850 – 1925) who did a ton of important work on EE, physics, and math, including diff. eq., using complex numbers for circuit analysis, and E&M.

In addition to looking like a guy you wouldn't want to mess with, he was pretty interesting — you can learn more about him online...

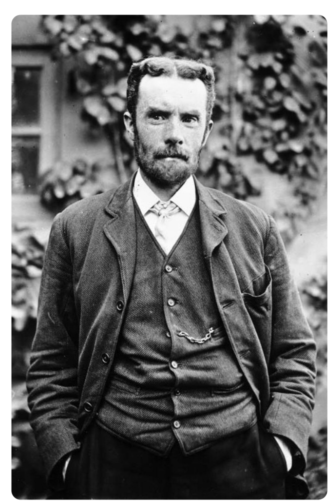


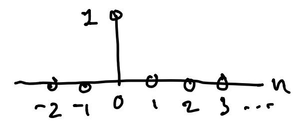
Image : Wikipedia

Main

The unit impulse function:

This one is easier to describe for DT signals first.

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



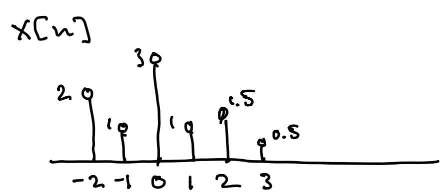
The unit impulse function or "delta function" is equal to 1 at time 0 and is 0 elsewhere. We will be seeing a lot of the unit impulse and its time shifts

$$\delta[n-m] = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

Why? Because we can write any DT signal as a sum of scaled & shifted impulse functions:

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\,\delta[n-m]$$

this looks sort of dumb & weird at first but be patient...



$$x[n] = 2\cdot\delta[n+2] + 1\cdot\delta[n+1] + 3\,\delta[n] + 1\cdot\delta[n-1] + 1.5\cdot\delta[n-2] + 0.5\,\delta[n-3]$$

An important property of the δ-function is that

$$\delta[n-m]\,\delta[n-k] = \begin{cases} 0 & m \neq k \\ \delta[n-m] & m = k \end{cases}$$

So we can use δ-functions to "pick out" elements of a DT signal:

$$\begin{aligned} x[n]\,\delta[n-m] &= \left(\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \right) \delta[n-m] \\ &= \sum_{k=-\infty}^{\infty} x[k] \underbrace{\delta[n-k]\,\delta[n-m]}_{\text{only } \neq 0 \text{ if } k=m} \\ &= x[m] \end{aligned}$$

Finally, we can write $\delta[n]$ in terms of shifted unit step functions:

$$\delta[n] = u[n] - u[n-1]$$

Try: verify this formula

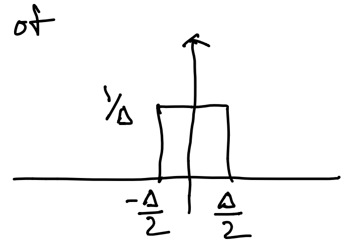
For CT signals the unit impulse $\delta(t)$, called the Dirac delta function, is a little trickier to define from a mathematical perspective. If we view the DT $u[n]$ as a "running sum" of $\delta[n]$: $u[n] = \sum_{m=-\infty}^{n} \delta[m]$

Then we can define $\delta(t)$ as the "function" that makes

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

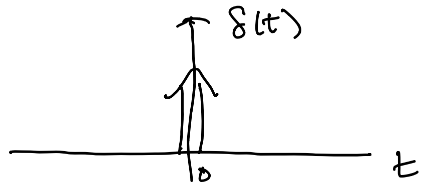
But this is weird: $u(t)$ jumps from 0 to 1 just after $t=0$. So somehow the area under $\delta(t)$ at $t=0$ has to be 1 !?!

Section 1.4.2 has a more formal treatment but one way to think about $\delta(t)$ is as a function that only is nonzero at $t=0$ but somehow packs an area 1 into there. Or as a limit of



as $\Delta \to 0$ the box gets taller & narrower

The δ-function is what is called a generalized function and we draw it like this:



with a $\Uparrow$ at $t=0$

We can also shift it: $\delta(t-3)$ is a $\uparrow$ at $t=3$

We can scale it in amplitude:

$\alpha\, \delta(t)$ has area α at $t=0$

Phil

Delta functions sound weird and intimidating at first but they are the right way to deal with idealizations like point masses or point charges in physics.

Man

When we integrate a signal multiplied by a shifted δ-function we can also "pull out" the function value

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-\tau)\,dt = x(\tau)$$

Similarly, we can formally define $\delta(t)$ as the derivative of $u(t)$:

$$\delta(t) = \frac{d}{dt}u(t) = \lim_{\Delta\to 0}\frac{u(t)-u(t-\Delta)}{\Delta}$$

compare this to the DT version:

$$\delta[n] = \frac{u[n]-u[n-1]}{1}$$

Try

Check yourself: Look at the learning objectives from the first page and see if you have learned those things. If not, try the other "Try" sections in this lecture to see if that helps.

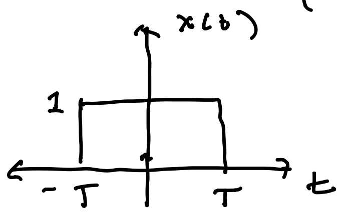
Main

Other Special functions

There are (at least) 3 other special functions we will be seeing a lot of in this class:

1) Rectangle or "boxcar" function:

$$x(t) = \begin{cases} 1 & |t| \leq T \\ 0 & t > T \end{cases}$$



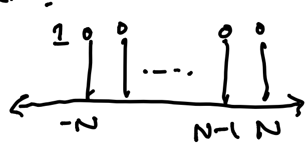
sometimes this is called rect(t) when $T = \frac{1}{2}$

!!!

Different books may have different notation for the rect(t) function. If you are looking at some other references or websites, be sure to check!

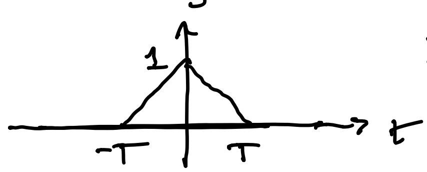
Main

The rectangle in discrete time is similar:



$$x[n] = \begin{cases} 1 & |n| \leq N \\ 0 & |n| < N \end{cases}$$

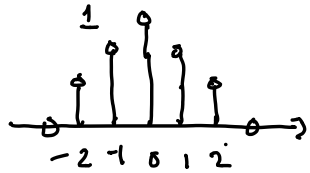
2) The triangle function



$$x(t) = \begin{cases} 1 + \frac{t}{T} & -T \leq t \leq 0 \\ 1 - \frac{t}{T} & 0 \leq t \leq T \end{cases}$$

$$= \begin{cases} 1 - \frac{|t|}{T} & -T \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

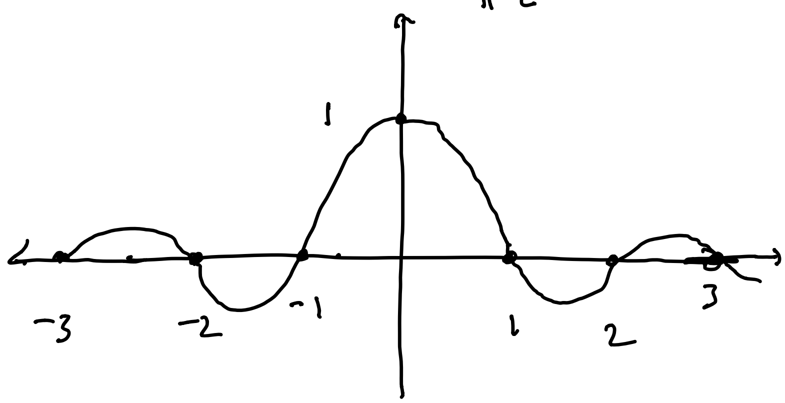
In DT we have the same thing:



$$x[n] = \begin{cases} 1 - \frac{|n|}{N} & |n| \le N \\ 0 & \text{otherwise} \end{cases}$$

3) The sinc (pronounced like "sink") function: (see p. 293-295)

$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$



crosses axis @ $t = k$, k integer, $k \neq 0$

$$\frac{\sin(k\pi)}{\pi k} = 0$$

$$\lim_{t \to 0} \frac{\sin(\pi t)}{\pi t} = 1$$

Try why is this true?

Try a Taylor series!