Continuous-Time Fourier Transforms (CTFTs)

Objectives:

- apply the CTFT definition to compute Fourier transforms / spectra from time domain signals and vice versa.

- use CTFT properties to derive transforms for complicated signals using simpler signals + signal operations

- understand the difference between CTFTs of periodic and aperiodic signals
- understand the sinc function and its properties (where does it cross 0, etc.)

Main

Motivating the CTFT

The CT Fourier Series (CTFS) gave us a way to write a CT periodic function as a LCCE:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

We can interpret each term $a_k e^{jk\omega_0 t}$ as saying that $x(t)$ contains a_k "amount" of the frequency $k\omega_0$. Since $x(t)$ has fundamental period $T = \frac{2\pi}{\omega_0}$, it only contains frequencies at integer multiples of ω_0

For aperiodic signals, we might have components at multiple frequencies, so if $x(t)$ is not periodic, then

$$x(t) - \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \neq 0$$

$$\text{when} \quad a_k = \frac{1}{T}\int_0^T x(t) e^{-jk\omega_0 t} dt$$

But we can still calculate the "amount" of frequency ω in $x(t)$ using something like

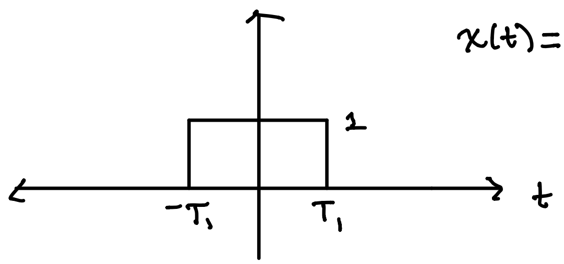
$$\int x(t) e^{-j\omega t} dt$$

But what should the limits of the integral be? The answer to this will give us a way to express the frequency content of general signals. This is the continuous time Fourier Transform.

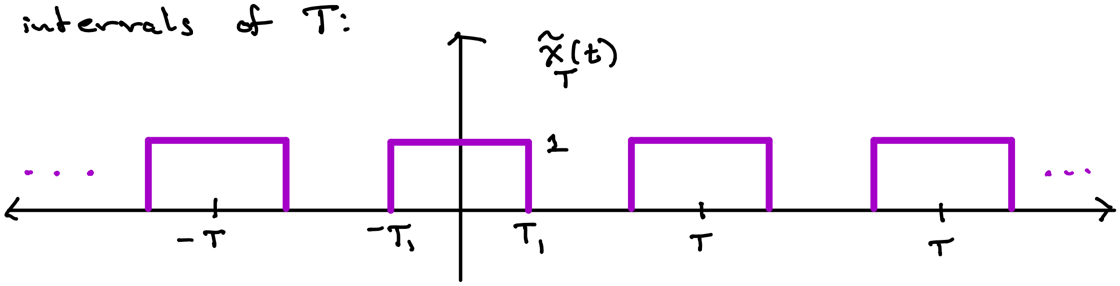
Back to square waves

One way to get a handle on the CTFT is to try and understand the frequency content of a rectangular pulse:

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & \text{otherwise} \end{cases}$$



We can extend this into a periodic signal with period $T > 2T_1$ by making copies of $x(t)$ at intervals of T:



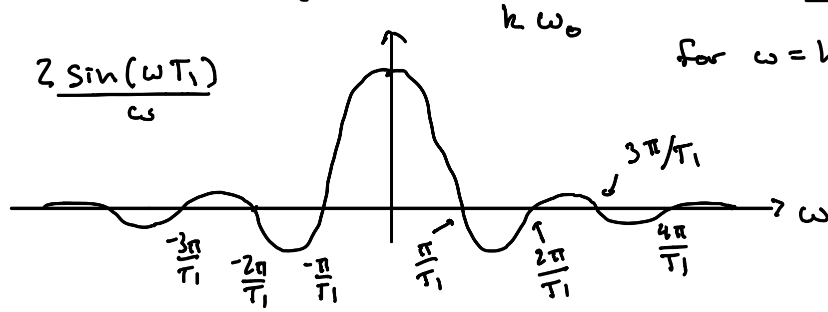
We found the CTFS for this signal:

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} \qquad \omega_0 = \frac{2\pi}{T}$$

Multiplying by T on both sides:

$$T a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0} = \frac{2\sin(\omega T_1)}{\omega}$$

for $\omega = k\omega_0$



Not

This function will appear a lot:

$$\operatorname{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}.$$

It's called the sinc (sounds like "sink") function and is $= 0$ when θ is an integer $\neq 0$.

Since $\omega_0 = \frac{2\pi}{T}$, if we take

$$\theta = 2k\frac{T_1}{T}$$

then

$$\operatorname{sinc}(\theta) = \frac{\sin(k\omega_0 T_1)}{k\omega_0 T_1}$$

so

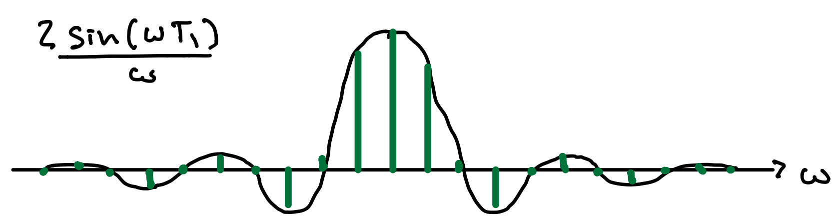
$$a_k = 2\frac{T_1}{T}\operatorname{sinc}(k\omega_0 T_1).$$

Main

Looking at the expression above, for different values of T (and hence different values of ω_0) we can look at the CTFS of $\tilde{x}_T(t)$ and see that it's just evenly spaced samples of

$$\frac{2\sin(\omega T_1)}{\omega}$$

at $\omega = k\frac{2\pi}{T}$, k going from $-\infty$ to ∞



(see Fig 4.2 in the book)

Since $\tilde{x}_T(t)$ is periodic with period T:

$$\tilde{x}_T(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t}$$

where

$$a_k = \frac{1}{T}\int_0^T \tilde{x}_T(t)\, e^{-jk\omega_0 t}\, dt$$

(since we just have to integrate over one period)

$$= \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}_T(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\underbrace{\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt}_{\text{call this } X(jk\omega_0)}$$

Since $\tilde{x}_T(t)$ is $x(t)$ between $-T/2$ and $\frac{T}{2}$

$$= \frac{1}{T} X(jk\omega_0)$$

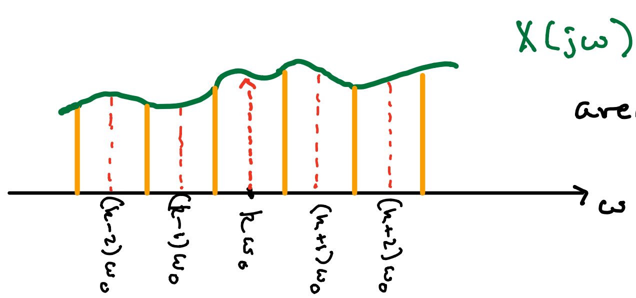
So now

$$\tilde{x}_T(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0)\, e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{2\pi}\, \omega_0\, X(jk\omega_0)\, e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0)\, e^{jk\omega_0 t}\, \omega_0$$

What happens as $T \to \infty$? The signal $\tilde{x}_T(t)$ gets more and more spread out so it "looks like" $x(t)$. Likewise, $\omega_0 \to 0$, so we can think of the terms in the sum as:



area in each strip is $\approx X(jk\omega_0)\,\omega_0$

So this sum becomes an integral as $\omega_0 \to 0$:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, dt \qquad \text{"synthesis"}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad \text{"analysis"}$$

Def.

Def: The continuous-time Fourier Transform (CTFT) of a signal $x(t)$ is

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt. \qquad \text{"analysis"}$$

The inverse Fourier transform is

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \qquad \text{"synthesis"}$$

Not.

We usually write lowercase $x(t)$ for time-domain signals and capital $X(j\omega)$ for frequency-domain signals.

The CTFT $X(j\omega)$ is also called the spectrum of $x(t)$. This terminology comes from light, where the spectrum refers to different colors of light waves. Color is associated to wavelength (period) and frequency.

Writing $X(j\omega)$ as a function of $j\omega$ and not ω is just the way we do it in engineering.

Main

The frequency ω is in radians/sec. What if we wanted Hertz ($= 1/\text{sec}$)?

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f}\, df$$

This is just from a change of variables:

$$\omega = 2\pi f$$

The effect is that the $\frac{1}{2\pi}$ goes away, which is kind of nice. The CTFT in radians (ω) is used more often in signal processing whereas Hertz (f) is more common in communications.

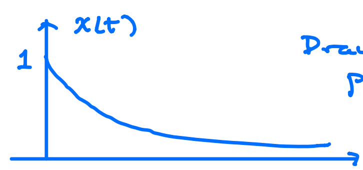
Some examples

Doing Fourier Transforms involves doing integrals. This might take some practice...

Ex

Examples:

1) $x_1(t) = e^{-at}\, u(t)$
$a > 0$



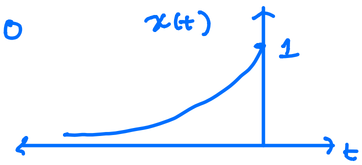
Draw a picture!

$$X_1(j\omega) = \int_{-\infty}^{\infty} e^{-at}\, u(t)\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \left[\frac{-1}{a+j\omega}\, e^{-(a+j\omega)t} \right]_{t=0}^{\infty} = \frac{1}{a+j\omega}$$

complex function

2) $x_2(t) = e^{at} u(-t) \quad a > 0$



$$X_2(j\omega) = \int_{-\infty}^{\infty} e^{at} u(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t} dt = \left[\frac{1}{a-j\omega} e^{(a-j\omega)t} \right]_{-\infty}^{0}$$

$$= \frac{1}{a-j\omega}$$ complex function

3) $x_3(t) = e^{-a|t|}$

This is just $x_1(t) + x_2(t)$.

Is the CTFT linear?

$$X_3(j\omega) = \int_{-\infty}^{\infty} (x_1(t) + x_2(t)) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt + \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} dt$$

$$= X_1(j\omega) + X_2(j\omega)$$

$$= \frac{1}{a+j\omega} + \frac{1}{a-j\omega}$$

$$= \frac{(a-j\omega) + (a+j\omega)}{a^2 + \omega^2}$$

$$= \frac{2a}{a^2+\omega^2}$$ real function!

Not

We will write

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

to indicate Fourier transform pairs.

Ex

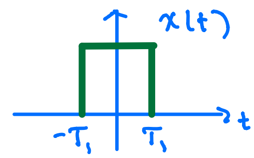
Examples:

1) $x(t) = \delta(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t} = e^{-j\omega(0)} = 1$$

So the unit impulse contains equal amounts of all frequencies

2) $x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| \geq T_1 \end{cases}$

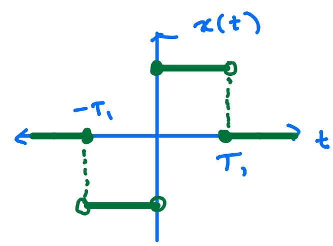


$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\, dt = \left[\frac{-1}{j\omega} e^{-j\omega t} \right]_{-T_1}^{T_1}$$

$$= -\frac{1}{j\omega} e^{-j\omega T_1} + \frac{1}{j\omega} e^{j\omega T_1}$$

also real! $\quad = \dfrac{2\sin(\omega T_1)}{\omega}$ we saw this one before...

3) $x(t) = \begin{cases} -1 & -T_1 < t < 0 \\ +1 & 0 \leq t < T_1 \\ 0 & \text{otherwise} \end{cases}$



$$X(j\omega) = \int_{-T_1}^{T_1} x(t)\, e^{-j\omega t}\, dt = \int_{-T_1}^{0} -e^{-j\omega t}\, dt + \int_{0}^{T_1} e^{-j\omega t}\, dt$$

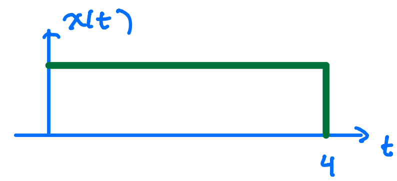
$$= \left[\frac{+1}{j\omega} e^{-j\omega t} \right]_{-T_1}^{0} + \left[\frac{-1}{j\omega} e^{-j\omega t} \right]_{0}^{T_1}$$

$$= \frac{1}{j\omega} - \frac{1}{j\omega} e^{+j\omega T_1} - \frac{1}{j\omega} e^{-j\omega T_1} + \frac{1}{j\omega}$$

$$= \frac{2}{j\omega} - \frac{2}{j\omega} \cos(\omega T_1)$$

$$= \frac{2(1-\cos(\omega T_1))}{j\omega}$$

4) $x(t) = u(t) - u(t-4)$



$$X(j\omega) = \int_0^4 e^{-j\omega t}\, dt$$

$$= \left[\frac{-1}{j\omega} e^{-j\omega t} \right]_0^4$$

$$= -\frac{1}{j\omega} e^{-j\omega 4} + \frac{1}{j\omega}$$

$$= \frac{1}{j\omega}\left(1 - e^{-j\omega 4}\right)$$

$$= e^{-j\omega 2} \frac{1}{j\omega}\left(e^{j\omega 2} - e^{-j\omega 2}\right)$$

$$= e^{-j\omega 2} \frac{2\sin(\omega 2)}{\omega}$$

Properties of the CTFT

The CTFT has many of the same properties as the CTFS. If $x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$ and $y(t) \xleftrightarrow{\mathcal{F}} Y(j\omega)$:

<u>Linearity</u>: $\alpha x(t) + \beta y(t) \xleftrightarrow{\mathcal{F}} \alpha X(j\omega) + \beta Y(j\omega)$

You can show this by plugging into the definition

<u>Time shift</u>: $x(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0} X(j\omega)$

See the example above.

<u>Conjugation</u>: $x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-j\omega)$

This means that if $x(t)$ is real ($x(t) = x^*(t)$) then $X(j\omega)$ has conjugate symmetry:

$$X(j\omega) = X^*(-j\omega)$$
$$X(-j\omega) = X^*(j\omega)$$

this also means (for real $x[n]$)

$$x_{even}(t) \xleftrightarrow{\mathcal{F}} \mathcal{Re}\{X(j\omega)\}$$

$$x_{odd}(t) \xleftrightarrow{\mathcal{F}} \mathcal{Im}\{X(j\omega)\}$$

Differentiation / Integration:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\,j\omega\, e^{j\omega t}\,d\omega$$

so $$\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{F}} j\omega X(j\omega)$$

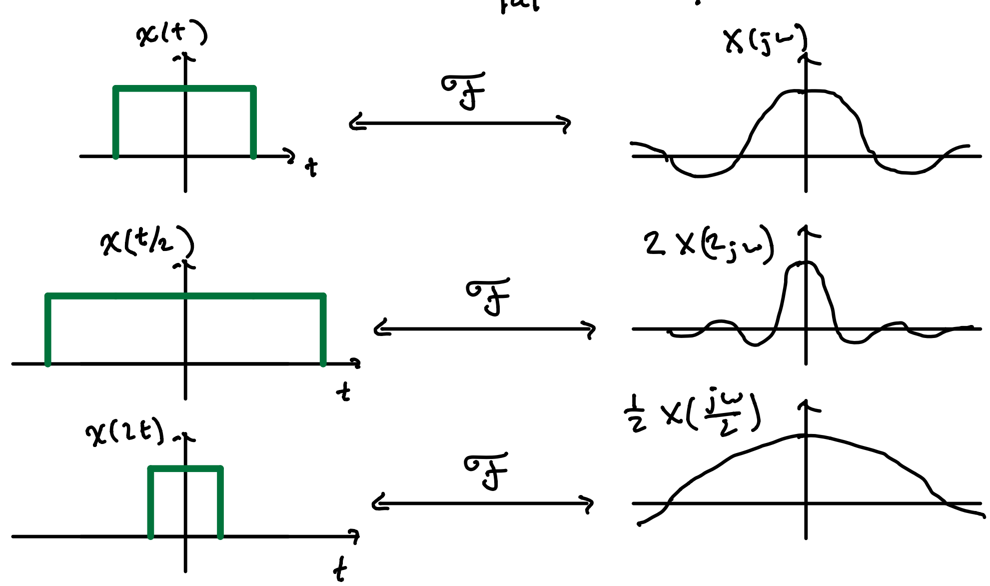
$$\int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{t} \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega\tau}\,d\omega\,d\tau$$

$$= \frac{1}{j\omega}X(j\omega) + \pi X(0)\,\delta(\omega)$$

(after a bit of calculus)

Time and frequency scaling:

$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|}X\left(\frac{j\omega}{a}\right)$$



Ex.

Examples

OW Problem 4.1 (a) Find the CTFT of

$$e^{-2(t-1)}u(t-1)$$

$$x(t) = e^{-2(t-1)}u(t-1)$$

$$= y(t-1) \quad \text{where} \quad y(t) = e^{-2t}u(t)$$

But we saw earlier

$$y(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2+j\omega} = Y(j\omega)$$

(time shift) $$y(t-1) \longleftrightarrow e^{-j\omega}Y(j\omega) = \frac{e^{-j\omega}}{2+j\omega}$$

OW Problem 4.6: Suppose $x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$. Find the CTFT of the following signals.

(a) $x_1(t) = x(1-t) + x(-1-t)$

For $x(1-t) = x(-(t-1))$ we first find

$t \to -t$ $$\mathcal{F}\{x(-t)\} = X(-j\omega)$$

(time reversal)

then

$t \to t-1$ $$\mathcal{F}\{x(-(t-1))\} = e^{-j\omega}X(-j\omega)$$

(time shift)

For $x(-1-t) = x(-(t+1))$:

$t \to -t$ $$\mathcal{F}\{x(-t)\} = X(-j\omega)$$

$t \to t+1$ $$\mathcal{F}\{x(-(t+1))\} = e^{+j\omega}X(-j\omega)$$

So: $$\mathcal{F}\{x_1(t)\} = \left(e^{-j\omega} + e^{j\omega}\right)X(-j\omega)$$

$$= 2\cos(\omega)\, X(-j\omega)$$

(b) $x_2(t) = x(3t-6)$

$= x(3(t-2))$

So $\mathcal{F}\{x(3t)\} = \frac{1}{3} X(j\frac{\omega}{3})$

$\mathcal{F}\{x(3(t-2))\} = \frac{1}{3} e^{-j2\omega} X(j\frac{\omega}{3})$

Why not do

$\mathcal{F}\{x(t-2)\} = e^{-j\omega 2} X(j\omega)$

$\mathcal{F}\{x(3(t-2))\} = \frac{1}{3} e^{-j\,2/3\,\omega} X(j\frac{\omega}{3})$

Because to apply the property

$y(3t) \longleftrightarrow \frac{1}{3} Y(j\frac{\omega}{3})$

we would get

$\mathcal{F}\{x(3t-2)\} = \frac{1}{3} e^{-j\,2/3\,\omega} X(j\frac{\omega}{3})$

and then to get $x(3t-6)$ we would need <u>another</u> delay

$t \to t - 4/3$:

$\mathcal{F}\{x(3(t-4/3)-2)\}$

$= \frac{1}{3} e^{-j\,2/3\,\omega} X(j\frac{\omega}{3}) \cdot e^{-j\,4/3\,\omega}$

$= \frac{1}{3} e^{-j2\omega} X(j\frac{\omega}{3})$

The moral of the story is to be careful about your variable transformations!

$x(t) \longrightarrow x(t-2)$

means $t \to t-2$

$x(t) \longrightarrow x(3t)$

means $t \to 3t$

(c) $x_3(t) = \frac{d^2}{dt^2} x(t-1)$

First delay:

$$y(t) = x(t-1) \longleftrightarrow e^{-j\omega} X(j\omega)$$

Then differentiate:

$$\frac{d}{dt} y(t) \overset{\mathcal{F}}{\longleftrightarrow} j\omega\, Y(j\omega)$$
$$= j\omega\, e^{-j\omega} X(j\omega)$$

$$\frac{d^2}{dt^2} y(t) \overset{\mathcal{F}}{\longleftrightarrow} (j\omega)^2 e^{-j\omega} X(j\omega)$$
$$= -\omega^2 e^{-j\omega} X(j\omega)$$

Main

The CTFT for periodic signals

We have focused on aperiodic signals to draw a contrast with the CTFS. But we can find the CTFT for periodic signals too.

Recall that we can express periodic signals as a LCCE via the CT Fourier Series:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{-jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$$

Since the CTFT is linear:

$$\mathcal{F}\{x(t)\} = \sum_{k=-\infty}^{\infty} a_k\, \mathcal{F}\{e^{-jk\omega_0 t}\}$$

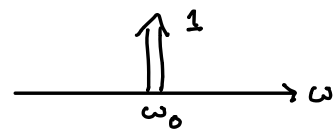
So we need to understand what the CTFT of a complex exponential

$$e^{j\omega_0 t}$$

works out to in the frequency domain.

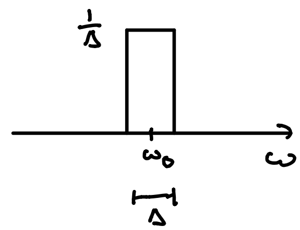
Another way to look at periodic signals would be to ask what time domain signal contains only frequency ω_0. That would correspond to

$$X(j\omega) = \delta(\omega - \omega_0)$$



As before, we can think of the δ-function as a limit as $\Delta \to 0$ of

$$X_\Delta(j\omega) = \begin{cases} \frac{1}{\Delta} & \omega_0 - \frac{\Delta}{2} \le \omega \le \omega_0 + \frac{\Delta}{2} \\ 0 & \text{otherwise} \end{cases}$$



then

$$x_\Delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_\Delta(j\omega)\, e^{j\omega t}\, d\omega$$

(applying limits on $X_\Delta(j\omega)$)
$$= \frac{1}{2\pi} \int_{\omega_0 - \frac{\Delta}{2}}^{\omega_0 + \frac{\Delta}{2}} \frac{1}{\Delta} e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi\Delta} \left[\frac{1}{jt} e^{j\omega t} \right]_{\omega_0 - \frac{\Delta}{2}}^{\omega_0 + \frac{\Delta}{2}}$$

$$= \frac{1}{2\pi\Delta} \left(\frac{1}{jt} \left(e^{j(\omega_0 + \Delta/2)t} - e^{j(\omega_0 - \Delta/2)t} \right) \right)$$

pulling out $e^{j\omega_0 t}$
$$= \frac{1}{2\pi\Delta} \frac{1}{jt} e^{j\omega_0 t} \left(e^{j\frac{\Delta}{2}t} - e^{-j\frac{\Delta}{2}t} \right)$$

"Eulerizing"
$$= \frac{1}{2\pi\Delta} \frac{1}{jt} e^{j\omega_0 t} \left(2j \sin\left(\frac{\Delta}{2} t\right) \right)$$

Taylor series for sin
$$= \frac{1}{\pi} e^{j\omega_0 t} \frac{1}{\Delta t} \left(\frac{\Delta}{2} t - \frac{1}{3!} \left(\frac{\Delta}{2} t \right)^3 + \cdots \right)$$

Δ's in the first term cancel...
$$= \frac{1}{2\pi} e^{j\omega_0 t} + \frac{1}{\pi} e^{j\omega_0 t} \left(-\frac{1}{3!} \frac{\Delta^2}{2^3} t^3 + \frac{1}{5!} \frac{\Delta^4}{2^5} t^5 + \cdots \right)$$

Then $\lim_{\Delta \to 0} x_\Delta(t)$

$$= \frac{1}{2\pi} e^{j\omega_0 t}$$

All other terms were $\Delta \cdot$ something so they $\longrightarrow 0$

But $\lim_{\Delta \to 0} X_\Delta(j\omega) = \delta(\omega - \omega_0)$

So

$$\frac{1}{2\pi} e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} \delta(\omega - \omega_0)$$

or (by linearity),

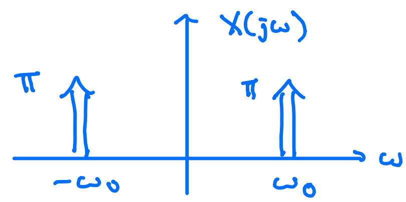
$$e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} 2\pi\, \delta(\omega - \omega_0)$$

Ex

So, for example:

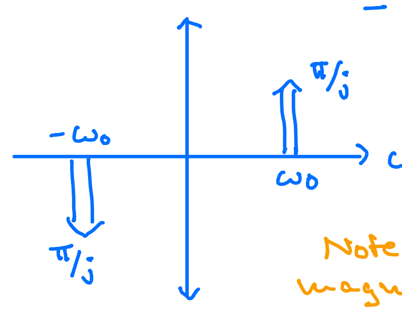
$$\cos(\omega_0 t) = \tfrac{1}{2} e^{j\omega_0 t} + \tfrac{1}{2} e^{-j\omega_0 t}$$

has $\mathcal{F}\{\cos(\omega_0 t)\} = \pi\, \delta(\omega - \omega_0) + \pi\, \delta(\omega + \omega_0)$



$$\sin(\omega_0 t) = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

has $\mathcal{F}\{\sin(\omega_0 t)\} = \frac{\pi}{j}\, \delta(\omega - \omega_0) - \frac{\pi}{j}\, \delta(\omega + \omega_0)$



Note this is not magnitude/phase.

Back to our CTFS:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

will have CTFT

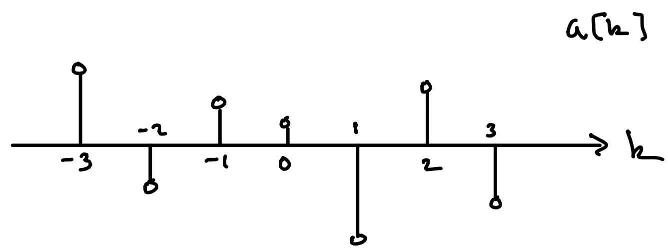
$$\mathcal{F}\{x(t)\} = \sum_{k=-\infty}^{\infty} a_k \, \mathcal{F}\{e^{jk\omega_0 t}\}$$

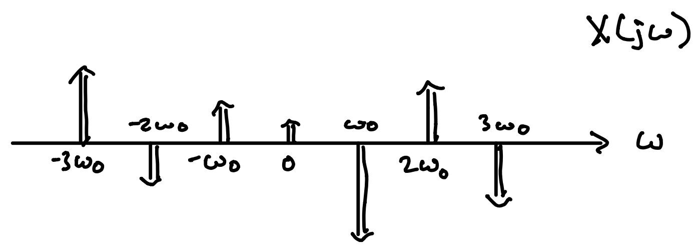
$$= \sum_{k=-\infty}^{\infty} a_k \, \delta(\omega - k\omega_0)$$

this turns our DT Fourier series signal

$$a[k] = a_k$$

into a train of impulse/δ-functions in the CT Fourier Transform spectrum:





The axis is labeled differently, and

$$a_{k_0} \delta[k - k_0] \longrightarrow a_{k_0} \delta(\omega - k_0\omega_0)$$

Duality, convolution, multiplication & signal properties

Another set of important CTFT properties relates to how signal properties in the time domain map to properties in the frequency domain. We saw some of these:

- If $x(t)$ is real, $x(t) = x^*(t)$, so

$$X(j\omega) = X^*(-j\omega) \quad \text{(conjugate symmetry)}$$

$$X(-j\omega) = X^*(j\omega)$$

To see this another way, look at synthesis:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\left(\int_{-\infty}^{0} X(j\omega)\, e^{j\omega t} d\omega + \int_{0}^{\infty} X(j\omega) e^{j\omega t} d\omega\right)$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} X(-j\omega)\, e^{-j\omega t} d\omega + \int_{0}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} \left(X^*(j\omega)\, e^{-j\omega t} + X(j\omega)\, e^{j\omega t}\right) d\omega$$

conjugates of each other

$$= \frac{1}{2\pi}\int_{0}^{\infty} 2\,\mathrm{Re}\left\{ X(j\omega)\, e^{j\omega t}\right\} d\omega$$

which is a <u>real</u> signal.

- If $x(t)$ is even, $x(t) = x(-t)$, so

$$X(j\omega) = X(-j\omega) \qquad \text{(real)}$$

Try

Try to derive the equivalent statements for imaginary and/or odd signals

Main

What about other properties?

Def

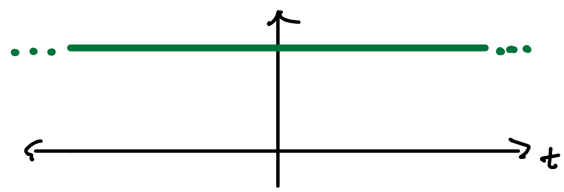
<u>DC component</u>: the frequency component at $\omega = 0$ is called the <u>DC</u> component:

$$X(j \cdot 0) = \int_{-\infty}^{\infty} x(t)\, e^{j \cdot 0 \cdot t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, dt$$

Def

So $X(0)$ is just the integral of $x(t)$. It is called the DC component because $e^{j \cdot 0 \cdot t} = 1$:



This looks like a direct current signal.

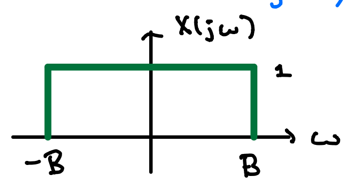
Similarly:

$$x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega 0}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, d\omega.$$

<u>Ex</u>: What is the DC component of $x(t) = \frac{\sin(Bt)}{\pi t}$?

For this sinc signal, $X(j\omega) = \begin{cases} 1 & |\omega| < B \\ 0 & \text{otherwise} \end{cases}$



Thus $X(0) = 1$.

This is a <u>lowpass</u> signal since it has <u>low</u> frequencies

Parseval's Relation:

Radians $$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

Hertz: $$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

This says that the total energy in a signal is the <u>same</u> in the time & frequency domains

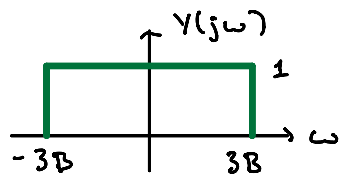
<u>Ex</u>: What is the energy of $x(t) = \dfrac{\sin(3Bt)}{2t}$?

First manipulate the function into a more familiar form:

$$x(t) = \frac{\sin(3Bt)}{\pi t} \cdot \frac{\pi}{2}$$

Let $y(t) = \dfrac{\sin(3Bt)}{\pi t}$ so $\mathcal{E}_x = \left(\frac{\pi}{2}\right)^2 \mathcal{E}_y$

We have $Y(j\omega) = \begin{cases} 1 & |\omega| < 3B \\ 0 & \text{otherwise} \end{cases}$



So

$$\mathcal{E}_y = \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(j\omega)|^2 \, d\omega$$

$$= \frac{1}{2\pi} \int_{-3B}^{3B} 1 \, d\omega = \frac{6B}{2\pi}$$

Thus $\mathcal{E}_x = \left(\frac{\pi}{2}\right)^2 \frac{6B}{2\pi} = \frac{3B\pi}{4}$.

There is a kind of symmetry in CTFT pairs: this is called <u>duality</u>:

$$x(t) = \begin{cases} 1 & |t| < T \\ 0 & \text{otherwise} \end{cases} \xleftrightarrow{\mathcal{F}} X(j\omega) = \frac{2\sin(\omega T)}{\omega}$$

$$x(t) = \frac{\sin(Bt)}{\pi t} \xleftrightarrow{\mathcal{F}} X(j\omega) = \begin{cases} 1 & |\omega| < B \\ 0 & \text{otherwise} \end{cases}$$

(see Figure 4.17 in the book)

This means that if some transformation in the <u>time domain</u> has an effect in the frequency domain (e.g. time shift by t_0 → multiplication by $e^{-j\omega t_0}$) then the same transformation in the <u>frequency domain</u> has a similar effect in the time domain.

<u>Frequency shift</u>: If $x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$ then what time domain signal corresponds to $X(j(\omega-\omega_0))$?

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j(\omega-\omega_0))\, e^{j\omega t}\, dt$$

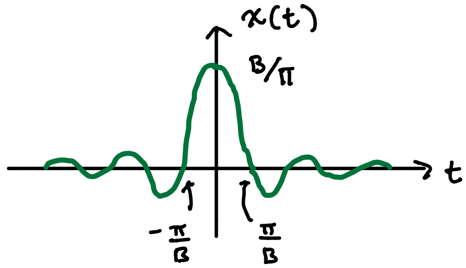
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, e^{j\omega_0 t}\, dt$$

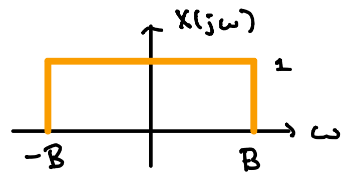
$$= e^{j\omega_0 t}\, X(j\omega)$$

So a frequency shift by ω_0 "up" is a multiplication by $e^{j\omega_0 t}$ in the time domain.

Ex

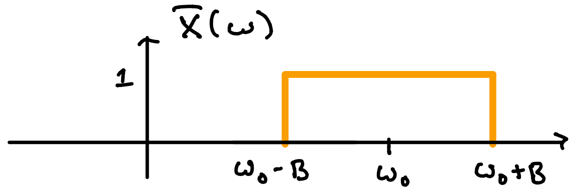
Ex. Suppose $x(t) = \dfrac{\sin(Bt)}{\pi t}$





Then $X(j\omega) = \begin{cases} 1 & |\omega| < B \\ 0 & \text{otherwise} \end{cases}$

If we look at $\bar{X}(\omega) = X(j(\omega - \omega_0))$:



This is a <u>bandpass</u> signal with frequencies in the band $[\omega_0 - B, \omega_0 + B]$

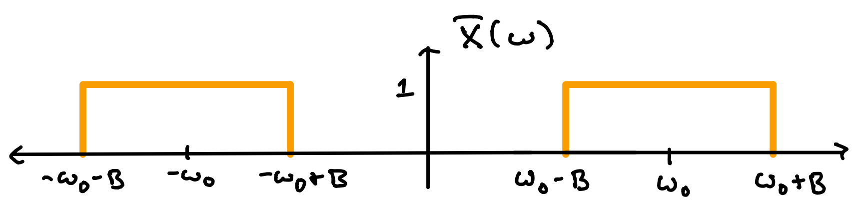
This corresponds to the signal

$$\bar{x}(t) = e^{j\omega_0 t} \frac{\sin(Bt)}{\pi t}$$

Note that this is a <u>complex</u> signal. To get a <u>real</u> signal we would need $\bar{X}(\omega)$ to be even.

So let's try

$$\tilde{X}(\omega) = X(j(\omega - \omega_0)) + X(j(\omega + \omega_0))$$



Now $\tilde{x}(t) = \left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) \dfrac{\sin(Bt)}{\pi t}$

$$= \frac{2\sin(Bt)}{\pi t} \cos(\omega_0 t).$$

This is a <u>real</u> signal: it is the original signal $x(t)$ <u>modulated</u> (multiplied) by a cosine at ω_0 radians/sec.

Main

This example shows how AM radio works: you take a <u>baseband</u> (lowpass) signal $x(t)$ whose spectrum lies between $-B$ and B and shift it up by <u>modulating</u> by a cosine signal at a frequency ω_0. The cosine is called the <u>carrier</u> and ω_0 is called the center frequency. AM stands for <u>amplitude</u> <u>modulation</u>: the signal $x(t)$ (e.g. a voice signal of a sports announcer) is the amplitude of the cosine. To make the signal easier to detect, real systems add a <u>DC component</u>:

$$y(t) = \underbrace{A}_{\substack{\text{gain}\\\text{from}\\\text{amplifier}}} \big(\underbrace{1}_{\substack{\text{DC}\\\text{offset}}} + \underbrace{k}_{\substack{\text{signal}\\\text{gain}}}\, x(t)\big) \underbrace{\cos(\omega_0 t)}_{\substack{\text{carrier}\\\text{signal}}}$$

We can analyze the spectrum $Y(j\omega)$ using $X(j\omega)$ and our CTFT properties.

Duality leads to other relations:

$$-jt\, x(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{d}{d\omega} X(j\omega)$$

$$-\frac{1}{jt}\, x(t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\omega} x(\varphi)\, d\varphi.$$

The most important property (and one of the things that, if nothing else, you should remember from this course, is the relationship between convolution and multiplication:

Convolution in time $\xleftrightarrow{\mathcal{F}}$ Multiplication in frequency

$$y(t) = (x * h)(t) \xleftrightarrow{\mathcal{F}} Y(j\omega) = X(j\omega)\,H(j\omega)$$

Multiplication in time $\xleftrightarrow{\mathcal{F}}$ Convolution in frequency

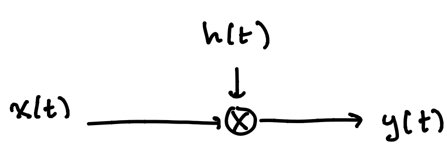
$$y(t) = x(t)\,h(t) \xleftrightarrow{\mathcal{F}} Y(j\omega) = \frac{1}{2\pi}(X * H)(j\omega)$$

or

$$Y(\omega) = (X * H)(\omega)$$

!!! Really — this is the most important fact.

Main We will start with the second version since it gives us a different way to understand frequency shifts.



This is a block diagram of the time domain

To see the property, we can plug into the definition.

Suppose $y(t) = \frac{1}{2\pi}\mathcal{F}^{-1}\{ X(j\omega) * H(j\omega)\}$. We want to show that $y(t) = x(t)\,h(t)$.

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\varphi)\, H(j(\omega-\varphi))\, d\varphi\; e^{j\omega t}\, d\omega$$

Def. of convolution

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\varphi)\, \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j(\omega-\varphi))\, e^{j\omega t}\, d\omega\; d\varphi$$

Change order of integration

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\varphi)\, e^{j\varphi t}\, \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} H(j(\omega-\varphi))\, e^{j(\omega-\varphi)t}\, d\omega}_{=\,h(t)}\, d\varphi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\varphi)\, e^{j\varphi t}\, h(t)\, d\varphi$$

$$= h(t)\,\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\varphi)\, e^{j\varphi t}\, d\varphi$$

$$= h(t)\, x(t)$$

Ex

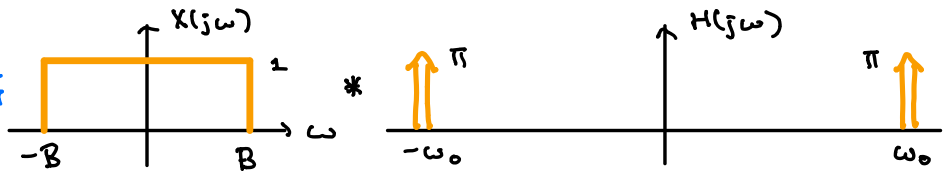
Example: consider $y(t) = \dfrac{\sin(Bt)}{\pi t}\cos(\omega_0 t)$

Then

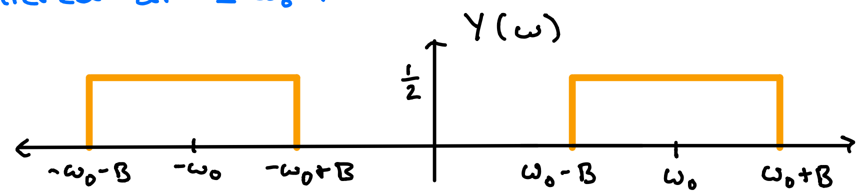
$$x(t) = \frac{\sin(Bt)}{\pi t} \xleftrightarrow{\mathcal{F}} X(j\omega) = \begin{cases} 1 & |\omega| < B \\ 0 & |\omega| > B \end{cases}$$

$$h(t) = \cos(\omega_0 t) \xleftrightarrow{\mathcal{F}} H(j\omega) = \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$$

$Y(j\omega) = \frac{1}{2\pi}$

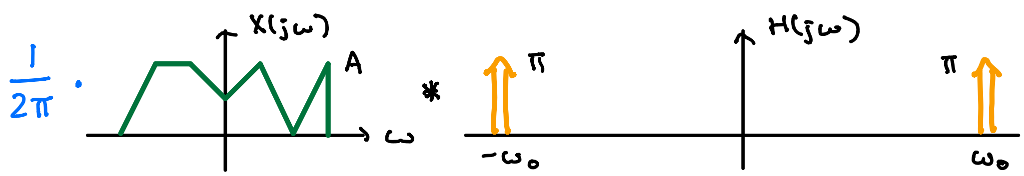


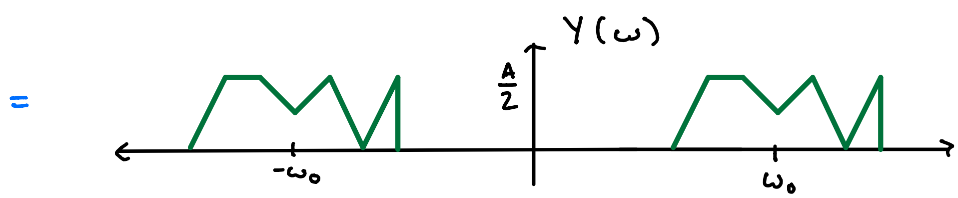
Remember that δ-functions make "copies" by convolution, so $Y(j\omega)$ is two copies of $X(j\omega)$ centered at $\pm\omega_0$:



The height is ½ since it's $\frac{1}{2\pi}(X * H)(j\omega)$

<u>Example</u>: this works for any signal $x(t)$ with CTFT $X(j\omega)$:





This corresponds to

$$y(t) = x(t)\cos(\omega_0 t)$$

Main

This gives us another way to understand the frequency shift property:

$$x(t)\,e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) * \delta(\omega-\omega_0)$$

$$= X(j(\omega-\omega_0))$$

What about convolution in the time domain?

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

definition of convolution
$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right) e^{-j\omega t}\, dt$$

expanding algebraically
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, h(t-\tau)\, e^{-j\omega(t-\tau)}\, d\tau\, dt$$

switching order of integration
$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau} \int_{-\infty}^{\infty} h(t-\tau)\, e^{-j\omega(t-\tau)}\, dt\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, H(j\omega)\, d\tau$$

doesn't depend on τ

$$= X(j\omega)\, H(j\omega)$$

Convolution in time = multiplication in frequency.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

For LTI systems

$$Y(j\omega) = X(j\omega)\, H(j\omega)$$

This is incredibly useful since it lets us understand (some) LTI systems without having to do convolutions!

Ex

<u>Example</u> (ideal lowpass filter). Suppose an LTI system has impulse response $h(t) = \frac{\sin(200\pi t)}{\pi t}$. Find the output $y(t)$ to the following inputs:

1) $x(t) = \dfrac{\sin(100\pi t)}{\pi t}$

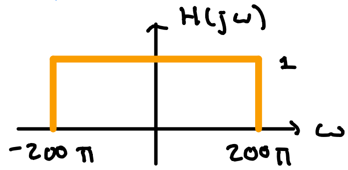
2) $x(t) = 2\cos(800\pi t)$

3) $x(t) = 5\sin(300\pi t) + 2\cos(100\pi t)$

4) $x(t) = e^{-2t} u(t)$

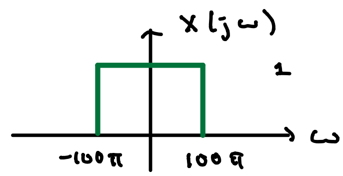
5) $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk70\pi t}$

All we have to do is compute the CTFT of $x(t)$ and then multiply by



This is called an ideal lowpass filter (LPF) because all frequencies $|\omega| < 200\pi$ pass through the system with gain 1 and all other frequencies with $|\omega| > 200\pi$ get "killed" by getting multiplied by $H(\omega) = 0$.

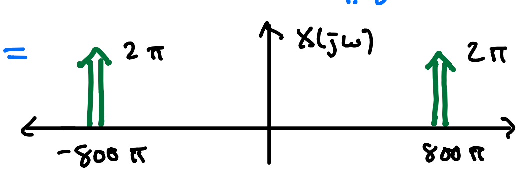
1) $X(j\omega) =$



So $Y(j\omega) = H(j\omega)X(j\omega) = X(j\omega)$

and $y(t) = \dfrac{\sin(100\pi t)}{\pi t}$

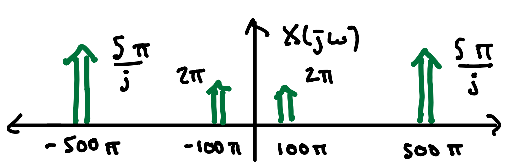
2) $X(j\omega) =$



$Y(j\omega) = X(j\omega)H(j\omega) = 0$

$y(t) = 0$ The filter kills $x(t)$.

3) $X(j\omega) =$



So $Y(j\omega) = X(j\omega)\, H(j\omega)$

$= 2\pi\, \delta(\omega - 100\pi) + 2\pi\, \delta(\omega + 100\pi)$

$y(t) = 2\cos(100\pi t)$

4) $X(j\omega) = \dfrac{1}{2+j\omega}$

$$Y(j\omega) = \begin{cases} \dfrac{1}{2+j\omega} & |\omega| < 200\pi \\ 0 & \text{otherwise} \end{cases}$$

5) $x(t)$ is given as a CTFS expansion into a LCCE: $\omega_0 = 70\pi$ and

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi\, a_k\, \delta(\omega - k70\pi)$$

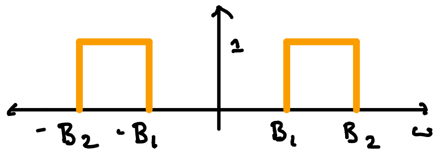
The LPF has a cutoff at 200π so only 5 terms are passed by the filter $H(\omega)$: $k = \pm 2, \pm 1$, and 0:

$$Y(j\omega) = \sum_{k=-2}^{2} 2\pi\, a_k\, \delta(\omega - k\,70\pi)$$

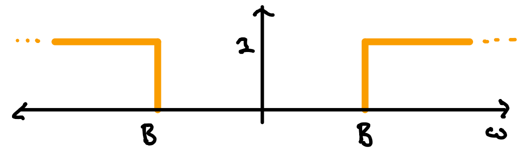
$$y(t) = a_{-2}\, e^{-j140\pi t} + a_{-1}\, e^{-j70\pi t} + a_0 + a_1\, e^{j70\pi t} + a_2\, e^{j140\pi t}$$

Main

In addition to the ideal LPF $x(t) = \dfrac{\sin(Bt)}{\pi t}$ we can define ideal <u>bandpass filters</u> and <u>highpass filters</u>:



bandpass



highpass

Using linearity we can get the impulse responses:

Lowpass: $h_{LPF}^{B}(j\omega) = \dfrac{\sin(Bt)}{\pi t}$

Highpass: $H_{HPF}^{B}(j\omega) = 1 - H_{LPF}^{B}(j\omega)$

$$h_{HPF}^{B}(t) = \delta(t) - \frac{\sin(Bt)}{\pi t}$$

Bandpass: $H_{BPF}^{B_1,B_2}(t) = H_{LPF}^{B_2}(j\omega) - H_{HPF}^{B_1}(j\omega)$

$$h_{BPF}^{B_1,B_2}(t) = \frac{\sin(B_2 t)}{\pi t} - \frac{\sin(B_1 t)}{\pi t}$$

Phil

Although we write down & analyze ideal filters, note that the impulse response of these systems is noncausal and so building ideal low/band/highpass filters is not possible. There are a lot of issues in filter design. The DSP course covers some more of these issues in more detail.

Mah

Magnitude and phase

If we have $x(t)$ going through an LTI system $h(t)$, what happens to the magnitude and phase of the signal?

$$Y(j\omega) = X(j\omega)\, H(j\omega)$$
$$= |X(j\omega)|\, |H(j\omega)|\, e^{j\measuredangle X(j\omega)}\, e^{j\measuredangle H(j\omega)}$$
$$= |X(j\omega)|\, |H(j\omega)|\, e^{j(\measuredangle X(j\omega) + H(j\omega))}$$

So:

$$|Y(j\omega)| = |X(j\omega)|\, |H(j\omega)|$$
$$\measuredangle Y(j\omega) = \measuredangle X(j\omega) + \measuredangle H(j\omega)$$

Ex

Example: $x(t) = e^{-2t} u(t)$ $\qquad h(t) = e^{-t} u(t)$

$$X(j\omega) = \frac{1}{2+j\omega} \qquad H(j\omega) = \frac{1}{1+j\omega}$$
$$= \frac{1}{4+\omega^2}(2-j\omega) \qquad = \frac{1}{1+\omega^2}(1-j\omega)$$
$$|X(j\omega)| = \frac{1}{\sqrt{4+\omega^2}} \qquad |H(j\omega)| = \frac{1}{\sqrt{1+\omega^2}}$$
$$\measuredangle X(j\omega) = -\tan\left(\frac{\omega}{2}\right) \qquad \measuredangle H(j\omega) = -\tan(\omega)$$

$$Y(j\omega) = \frac{1}{(2+j\omega)(1+j\omega)} = \frac{1}{(2-\omega^2)+j3\omega}$$

$$|Y(j\omega)| = \left(\frac{1}{(4+\omega^2)(1+\omega^2)}\right)^{1/2} = \frac{1}{\sqrt{4+5\omega^2+\omega^4}}$$

$$\measuredangle Y(j\omega) = -\tan\left(\frac{\omega}{2}\right) - \tan(\omega)$$

Main

Bandpass filters get used a lot in audio signal processing and communications. With the tools we have now we can understand several systems such as equalizers and radio-frequency (RF) communication systems.