Z - Transforms

Objectives: You should be able to

- compute the z-transform of DT signals and the region of convergence, or ROC.
- draw and interpret pole-zero diagrams to find the ROC
- explain the connection between the z-transform and the DTFT
- write the transfer function for a system from input-output responses
- use partial-fraction expansion to invert the z-transform for rational functions
- identify signal and system properties from the z-transform and ROC
- find the DT signals corresponding to a z-transform using different ROCs
- use z-transform properties to simplify calculations
- use the multiplication / convolution property to find the output of LTI systems

Main

If we are going to build DT signal processing methods for CT signals, we might need to understand DT systems in more detail. Our previous work left some questions unanswered, such as how to determine if a system has a stable and causal inverse. The z-transform will help answer those questions.

Def

<u>Def</u>. The z-transform of a DT signal $x[n]$ is the polynomial

$$\mathcal{Z}\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

A causal signal is also called <u>right-sided</u> and

$$X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n}$$

$$= x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

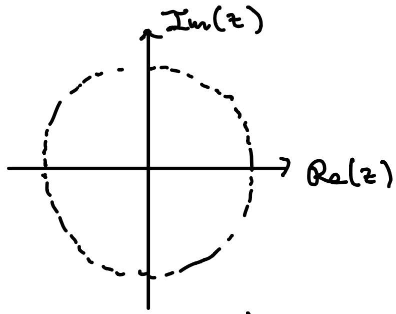
An anti-causal signal is also called left-sided and

$$X(z) = \sum_{n=-\infty}^{0} x[n]\, z^{-n}$$

$$= \sum_{n=0}^{\infty} x[-n]\, z^{n}$$

$$= x[0] + x[-1]z + x[-2]z^{2} + \cdots$$

The z-transform $X(z)$ of $x[n]$ is a polynomial in the variable z, where $z \in \mathbb{C}$. So this is a function on the complex plane



If we look at a right-sided (causal) signal we can think of it as a polynomial in z^{-1}. One thing about finite degree polynomials is that they factorize:

$$X(z) = x[0] + x[1]\, z^{-1} + \cdots + x[n]\, z^{-m}$$

$$= a_0 \prod_{i=1}^{m} (z - a_i)$$

The points $a_1 \cdots a_m$ are the roots of the polynomial $X(z)$: $X(a_i) = 0$ so we also call them <u>zeroes</u> of $X(z)$.

Introducing the Region of Convergence

It turns out that there can be more than one signal with the same z-transform:

Ex

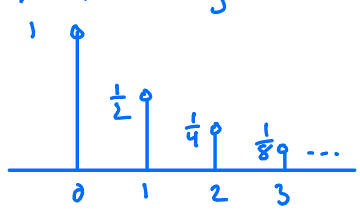
<u>Example</u>: Consider the two sequences

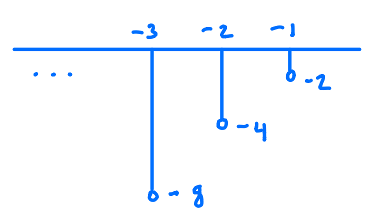
$x_1[n] = \frac{1}{2^n} u[n]$

$x_2[n] = -\frac{1}{2^n} u[-n-1]$

$x_1[n]$ is right-sided

$x_2[n]$ is left-sided





What about their z-transforms?

$$X_1(z) = \sum_{n=0}^{\infty} \frac{1}{2^n} z^{-n} \qquad X_2(z) = \sum_{n=-\infty}^{-1} -\frac{1}{2^n} z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{2} z^{-1}\right)^n \qquad = \sum_{n=1}^{\infty} -(2z)^n$$

$$= \frac{1}{1-\frac{1}{2}z^{-1}} \qquad = 1 - \sum_{n=0}^{\infty} (2z)^n$$

$$= \frac{z}{z-\frac{1}{2}} \qquad = 1 - \frac{1}{1-2z}$$

$$= \frac{z}{z - 1/2}$$

So $X_1(z)$ and $X_2(z)$ are the same, algebraically...

What's going on? It turns out that specifying the polynomial isn't enough — we have to describe the set of z for which $X(z)$ converges. This is (unsurprisingly) called the region of convergence (ROC) for the z-transform.

Looking at our example above, where does

$$\sum_{n=0}^{\infty} \frac{1}{2} z^{-n} \text{ not converge to } \frac{z}{z-\frac{1}{2}} \; ?$$

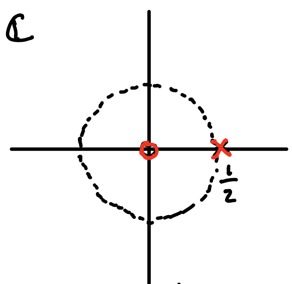
That happens when the denominator is 0, or when $z = \frac{1}{2}$. We say that $X(z)$ has a <u>pole</u> at $z = 1/2$. In general, a lot of the z-transforms that we will see will be rational functions with polynomials in the numerator and denominator:

$$X(z) = \frac{\alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha^m z^m}{\beta_0 + \beta_1 z + \beta_2 z^2 + \cdots + \beta^k z^h}$$

$$= C_0 \frac{\prod_{i=1}^{m} (z - a_i)}{\prod_{j=1}^{k} (z - b_i)}$$

zeros of $X(z)$

poles of $X(z)$



pole-zero diagram for example

We can draw the poles and zeroes of $X(z)$ on the complex plane using "o" for zeroes and "x" for poles.

For every pole b_i we draw a circle at radius $|b_i|$. This splits up the complex plane into a series of concentric rings. The ROC is determined by those rings.

For our example, $x_1[n] \xrightarrow{\mathcal{Z}} \frac{z}{z - 1/2}$ for $|z| > \frac{1}{2}$

What about $x_2[n]$?

$$\sum_{n=-\infty}^{-1} -\frac{1}{2^n} z^{-n}$$ will converge if z is large enough: $\frac{1}{2^n}$ is blowing up for negative n, so z^{-n} has to be shrinking fast enough to compensate: $|z| < \frac{1}{2}$.

So for our example:

$$X_1(z) = \frac{z}{z - 1/2} \quad \text{with ROC } |z| > 1/2$$

$$X_2(z) = \frac{z}{z - 1/2} \quad \text{with ROC } |z| < 1/2$$

Not

To fully specify a z-transform we have to give the formula for $X(z)$ <u>and</u> the ROC.

⟶ when we talk about z-transform properties, a transformation (e.g. upsampling) affects both $X(z)$ <u>and</u> the ROC.

!!!

The ROC issue makes z-transforms quite a bit different than the Fourier transforms we have seen so far.

Main

For signals that are complex exponentials, we can write the z-transform:

$$x[n] = a^n u[n] \xleftrightarrow{\mathcal{Z}} X(z) = \frac{z}{z-a} \quad \text{ROC } |z| > |a|$$

for $|a| < 1$

The z-transform is linear:

$$\mathcal{Z}\{\alpha x[n] + \beta y[n]\} = \sum_{n=-\infty}^{\infty} (\alpha x[n] + \beta y[n]) z^{-n}$$

$$= \alpha \sum_{n=-\infty}^{\infty} x[n] z^{-n} + \beta \sum_{n=-\infty}^{\infty} y[n] z^{-n}$$

$$= \alpha X(z) + \beta Y(z)$$

What about the ROC? It is $\text{ROC}(x[n]) \cap \text{ROC}(y[n])$

Why? Because we need <u>both</u> $X(z)$ and $Y(z)$ to converge.

Ex

<u>Example</u> $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n-2]$

Then $\left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{\mathcal{Z}} \dfrac{z}{z-\frac{1}{2}}$ ROC $|z| > \frac{1}{2}$

$$\mathcal{Z}\left\{\left(\tfrac{1}{3}\right)^n u[n-2]\right\} = \sum_{n=2}^{\infty} \left(\tfrac{1}{3}\right)^n z^{-n}$$

$$= -1 - \frac{1}{3} z^{-1} + \sum_{n=0}^{\infty} \left(\tfrac{1}{3}\right)^n z^{-n}$$

$$= -1 - \frac{1}{3} z^{-1} + \frac{z}{z - 1/3}$$

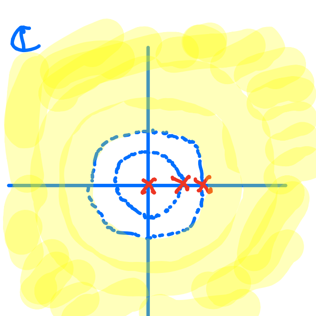
$$= \frac{-\left(1 + \frac{1}{3} z^{-1}\right)\left(z - 1/3\right) + z}{z - 1/3}$$

$$= \frac{-z + \frac{1}{3} - \frac{1}{3} + \frac{1}{3} z^{-1} + z}{z - 1/3}$$

$$= \frac{1/3}{z\left(z - 1/3\right)}$$ poles at $z = 0$ and $z = \frac{1}{3}$

ROC $|z| > \frac{1}{3}$

Then $\mathcal{Z}\{x[n]\} = \dfrac{z}{z - \frac{1}{2}} + \dfrac{1/3}{z(z - 1/3)}$ ROC $|z| > 1/2$



ROC shown in yellow

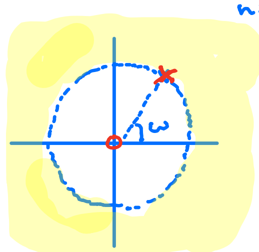
$$= \frac{z^2(z - 1/3) + 1/3(z-2)}{z(z - 1/2)(z - 1/3)}$$

$$= \frac{z^3 - 1/3 z^2 + 1/3 z - 2/3}{z(z - 1/2)(z - 1/3)}$$

$= ???$ (can find using MATLAB)

Example: $x[n] = a^n e^{j\omega n} u[n]$

$$X(z) = \sum_{n=0}^{\infty} (a e^{j\omega} z^{-1})^n = \frac{z}{z - a e^{j\omega}}$$



ROC: $|z| > |a e^{j\omega}| = |a|$

Main

If we can find the z-transform of LCCEs we can get the z-transforms of periodic signals.

If we set $z = e^{j\omega}$ then

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = X(e^{j\omega}) \text{ the DTFT}$$

The points $\{e^{j\omega} : \omega \in [0, 2\pi]\}$ are on the unit circle in $\mathbb{C}$. So if the ROC for $X(z)$ includes the unit circle, then the DTFT of $x[n]$ exists.

Try

Find the z-transform, ROC, and pole-zero diagram for $a^n \cos(\omega_0 n) u[n]$, $b^n \sin(\omega_0 n)$, and

$$a^n \cos(\omega_0 n) + b^n \sin(\omega_0 n)$$

Main

Characterizing the ROC

For a general signal $x[n]$, what does the ROC look like and what properties does it have?

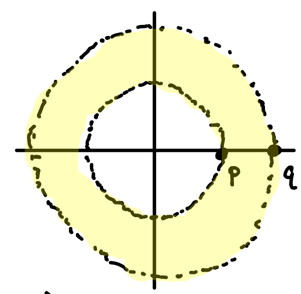
We can divide these properties up into 3 categories: general, right/left sided, and rational.

General: Suppose $x[n]$ has z-transform $X(z)$.

1) The ROC is an annulus (ring) in the complex plane, where the inner radius may be 0 and outer radius may be ∞.
2) The ROC contains no poles.
3) If $x[n]$ is finite length, then the ROC is one of the following:

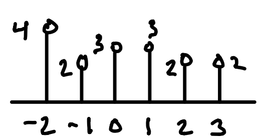
$|z| \geq 0$ (the whole complex plane)
$|z| > 0$ (e.g. pole @ $z = 0$)
$|z| < \infty$ (e.g. pole @ $z = \infty$)
$0 < |z| < \infty$ (e.g. poles @ $z=0$ and $z=\infty$)

Property 1) means that the ROC in general is of the form $p < |z| < q$



Property 2) makes sense because $X(z)$ blows up (does not converge) at poles, so the region of convergence should not have any poles.

Property 3 comes from the algebraic structure of $X(z)$:



$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

$$X(z) = \sum_{k=-\infty}^{\infty} x[k]\, z^{-k}$$

$-k$ in both places

$$X(z) = 4z^2 + 2z + 3 + 3z^{-1} + 2z^{-2} + 2z^{-3}$$

blows up for $z = 0$ and $|z| = \infty$

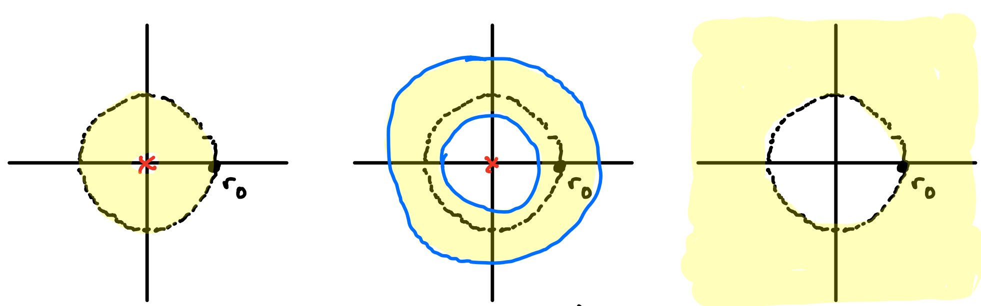
so the ROC is $0 < |z| < \infty$

These three properties tell us about the general shape of the ROC. If we know more about $x[n]$ we can say more.

4) If $x[n]$ is right-sided (causal) and the circle $|z| = r_0$ is in the ROC, then the ROC contains all finite z such that $|z| > r_0$.

5) If $x[n]$ is left-sided (anticausal) and the circle $|z| = r_0$ is in the ROC, then $0 < |z| < r_0$ is also in the ROC.

6) If $x[n]$ is two-sided (neither causal nor anticausal) and $|z| = r_0$ is in the ROC, then the ROC is a ring including the circle $|z| = r_0$



left-sided includes inside the circle

two-sided is a ring around the circle

right-sided includes outside the circle

Again, these properties follow from looking at $X(z)$. If $X(z)$ converges for $|z| = r_0$

$$X(z) = \sum_{n=-\infty}^{0} x[n] z^{-n}$$ blows up for large $|z|$
$\rightarrow$ converges for smaller $|z| < r_0$

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$ blows up for small $|z|$
$\rightarrow$ converges for larger $|z| > r_0$

For two-sided sequences we saw that there are poles at $z = 0$ and $z = \infty$

To connect back to our earlier example where

$$\mathcal{Z}\left\{ \left(\tfrac{1}{2}\right)^n u[n] \right\} = \frac{z}{z - 1/2} = \mathcal{Z}\left\{ -\left(\tfrac{1}{2}\right)^n u[-n-1] \right\}$$

we can see that one is right-sided and the other left-sided, so they have different ROCs.

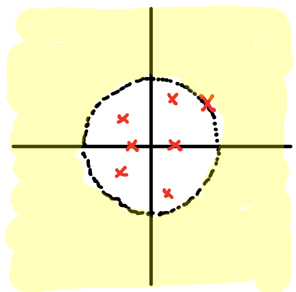
The last set of properties have to do with <u>rational</u> $X(z)$. For these z-transforms we can factorize to get the zeros in the numerator and poles in the denominator:

$$X(z) = C_0 \frac{\prod_{i=1}^{m} (z - a_i)}{\prod_{j=1}^{k} (z - b_j)}$$

7) If $X(z)$ is rational then $X(z)$ either has a pole at ∞ or the ROC contains ∞.

   $\Rightarrow$ If $m > k$ then the numerator blows up as $z \to \infty$ so there will be a pole there.

8) If $X(z)$ is rational and right-sided with poles $\{b_i\}$ such that $|b_1| \le |b_2| \le \cdots \le |b_k|$ then the ROC is $|z| > |b_k|$ (the region outside the largest pole). If $x[0] = 0$ then the ROC contains $|z| = \infty$.

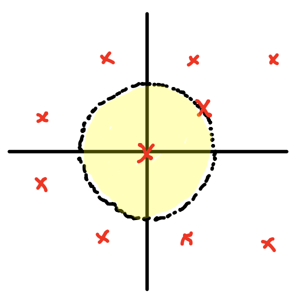


If $x[n]$ is <u>strictly</u> causal then

$$X(z) = x[1]\,z^{-1} + x[2]\,z^{-2} + \cdots$$

which goes to 0 when $z \to \infty$

9) If $X(z)$ is rational and left-sided with poles $\{b_i\}$ such that $|b_1| \le |b_2| \le \cdots \le |b_k|$ then the ROC is $0 < |z| < |b_1|$ (the region inside the smallest pole). If $x[0] = 0$ then the ROC contains $|z| = 0$.

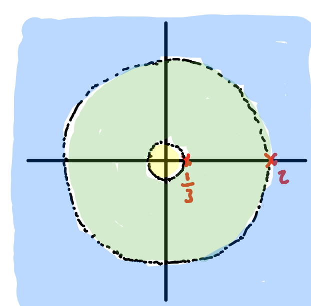


If $x[n]$ is <u>strictly</u> anticausal then $X(z) = x[-1]\,z + x[-2]\,z^2 + \cdots$ which is $= 0$ for $z = 0$.

Ex

<u>Example</u> Given an $X(z)$ we would like to find the different ROCs that are possible:

$$X(z) = \frac{1}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - 2z^{-1}\right)}$$



The first thing to do is draw the pole-zero diagram. There are two poles at

$$\left.\begin{aligned} z^{-1} &= 3 \Rightarrow z = \tfrac{1}{3} \\ z^{-1} &= \tfrac{1}{2} \Rightarrow z = 2 \end{aligned}\right\}$$

This shows why writing $X(z)$ in terms of z^{-1} can be useful...

So there are three possible

regions to consider:

inside the circle $|z| < 1/3$ left-sided

between the circles $1/3 < |z| < 2$ two-sided

outside the circle $|z| > 2$ right-sided

So we can identify the ROCs but how can we find the three different $x[n]$ signals corresponding to these three ROC's?

Main

Partial fraction expansion and inverse transforms

How do we invert the z-transform? In general, the answer is ugly integration.

If $x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$

$$x[n]r^{-n} \xleftrightarrow{\mathcal{F}} \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n}$$

$$= X(re^{j\omega})$$

So if we think of $z = re^{j\omega}$ then $X(z) = X(re^{j\omega})$

The problem is that $r = 1$ might not be in the ROC. So

$$x[n]r^{-n} = \mathcal{F}^{-1}\{X(re^{j\omega})\}$$

$$x[n] = r^n \frac{1}{2\pi}\int_0^{2\pi} X(re^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_0^{2\pi} X(re^{j\omega})\,(re^{j\omega})^n\, d\omega$$

If we fix r, this is an integral around a circle of radius r in the complex plane.

Let's do the variable substitution:

$$z = r e^{j\omega}$$

$$dz = r j e^{j\omega} \, d\omega$$

$$= j z \, d\omega \quad \Rightarrow \quad d\omega = \frac{1}{j} z^{-1} dz$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

integral along a path $\Rightarrow$ circle of radius r, going counter-clockwise

This doesn't seem to have made things better: evaluating path integrals is often quite messy / tough.

What to do?

We know how to invert

$$\frac{z}{z-a} = \frac{1}{1-az^{-1}}$$

So for rational $X(z)$ maybe we can write it out as a sum of things we know how to invert.

Ex

Let's expand the set of things we know how to invert:

Suppose $y[n] = x[n-n_0]$ (a delay by n_0)

then

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n-n_0] z^{-n}$$

$$= z^{-n_0} \sum_{n=-\infty}^{\infty} x[n] z^{-n} = z^{-n_0} X(z).$$

So

$$a^{n-n_0}u[n-n_0] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-n_0}}{1-az^{-1}}$$

Thus by linearity we have that

$$\sum_{\ell=M}^{N} c_\ell\, a^{n-\ell}\, u[n-\ell] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{\sum_{\ell=M}^{N} c_\ell\, z^{-\ell}}{1-az^{-1}}$$

So any polynomial in the numerator can also be inverted. If you look at the LHS you might see that it looks a lot like a convolution. More on that later.

Main

So what we want is to break apart $X(z)$ using partial fraction expansion.

Start with two terms:

$$\frac{1}{(1-b_1z^{-1})(1-b_2z^{-1})} = \frac{c_1}{1-b_1z^{-1}} + \frac{c_2}{1-b_2z^{-1}}$$

$$1 = c_1(1-b_2z^{-1}) + c_2(1-b_1z^{-1})$$

$$1 = c_1 + c_2 - (c_1b_2 + c_2b_1)z^{-1}$$

$$\Rightarrow \begin{matrix} c_1 + c_2 = \\ c_1b_2 + c_2b_1 = 0 \end{matrix} \quad \Rightarrow \quad c_1 = -\frac{b_1}{b_2}c_2$$

$$c_2(1 - b_1/b_2) = 1$$

$$\begin{cases} C_2 = \dfrac{1}{1-b_1/b_2} = \dfrac{b_2}{b_2-b_1} \\[2ex] C_2 = \dfrac{-b_1}{b_2-b_1} = \dfrac{b_1}{b_1-b_2} \end{cases}$$

How does this work? Multiply out denominators and match up coefficients of powers of z (or z^{-1}).

$$\frac{1-a_1 z^{-1}}{(1-b_1 z^{-1})(1-b_2 z^{-1})} = \frac{C_1}{1-b_1 z^{-1}} + \frac{C_2}{1-b_2 z^{-1}}$$

$$1 - a_1 z^{-1} = (C_1 + C_2) - (C_1 b_2 + C_2 b_1) z^{-1}$$

$$\Rightarrow \begin{array}{l} C_1 + C_2 = 1 \\ C_1 b_2 + C_2 b_1 = a_1 \end{array} \quad \Rightarrow \quad C_1 = \frac{a_1}{b_2} - \frac{b_1}{b_2} C_2$$

$$\frac{a_1}{b_2} + (1 - b_1/b_2) C_2 = 1$$

$$C_2 = \frac{1 - a_1/b_2}{1 - b_1/b_2} = \frac{b_2 - a_1}{b_2 - b_1}$$

$$C_1 = \frac{a_1}{b_2} - \frac{b_1}{b_2} \frac{b_2 - a_1}{b_2 - b_1}$$

$$= \frac{a_1 b_2 - a_1 b_1 - b_1 b_2 + a_1 b_1}{b_2 (b_2 - b_1)}$$

$$= \frac{b_2 (a_1 - b_1)}{b_2 (b_2 - b_1)}$$

$$= \frac{a_1 - b_1}{b_2 - b_1}$$

Now that we have a few examples, what about

$$\frac{P(z)}{\prod_{i=1}^{m} (1 - b_i z^{-1})} = \sum_{i=1}^{m} \frac{C_i}{1 - b_i z^{-1}} \quad ?$$

Multiply by $(1 - b_j z^{-1})$:

$$\frac{P(z)}{\prod_{i \neq j} (1 - b_i z^{-1})} = C_j + \sum_{i \neq j} C_i \frac{1 - b_j z^{-1}}{1 - b_i z^{-1}}$$

If all of the $\{b_i\}$ are distinct, plug in $z = b_j$:

$$\frac{P(b_j)}{\prod_{i \neq j} (1 - b_i/b_j)} = C_j \qquad \text{since the sum} \to 0.$$

What if we have multiple identical poles?

→ a bit more complicated. See Appendix A.3.

Ex

Example: Find the signals with z-transform

$$X(z) = 2\frac{1 - 1/3\, z^{-1}}{1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}} .$$

Step 1: get it into factorized form with zeros on the numerator and poles on the denominator:

$$X(z) = 2\frac{(1 - 1/3\, z^{-1})}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}$$ zero at $\frac{1}{3}$, poles at $\frac{1}{2}, \frac{1}{4}$

Step 2: find the partial fraction expansion:

$b_1 = \frac{1}{2}$, $b_2 = \frac{1}{4}$, $P(z) = 1 - 1/3\, z^{-1}$

$$C_1 = \frac{P(1/2)}{(1-\frac{1}{4}\, 2)} = \frac{1-2/3}{1-1/2} = \frac{1/3}{1/2} = 2/3$$

$$C_2 = \frac{P(1/4)}{(1-\frac{1}{2}\, 4)} = \frac{1-4/3}{1-2} = \frac{-1/3}{-1} = 1/3$$

So

$$X(z) = \frac{4/3}{1-\frac{1}{2}z^{-1}} + \frac{2/3}{1-\frac{1}{4}z^{-1}}$$

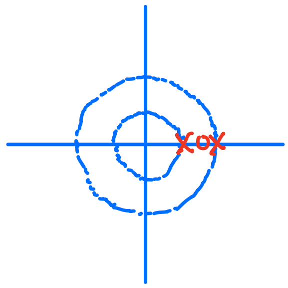
verify:

$$\frac{4/3\,(1-1/4\,z^{-1}) + 2/3\,(1-1/2\,z^{-1})}{(1-1/2\,z^{-1})(1-1/4\,z^{-1})}$$

$$= \frac{4/3 - 1/3\,z^{-1} + 2/3 - 1/3\,z^{-1}}{(1-1/2\,z^{-1})(1-1/4\,z^{-1})}$$

$$= \frac{2 - 2/3\,z^{-1}}{(1-\frac{1}{2}z^{-1})(1-1/4\,z^{-1})}$$ ✓✓

Step 3: Identify potential ROC's.



$\frac{1}{1-\frac{1}{2}z^{-1}}$: $|z| > 1/2$ right-sided
or
$|z| < 1/2$ left-sided

$\frac{1}{1-\frac{1}{4}z^{-1}}$: $|z| > 1/4$ right-sided
or
$|z| < 1/4$ left-sided

The ROC for $X(z)$ is the intersection.

4 combinations: do all work?

1. $|z| > 1/2 \;\cap\; |z| > 1/4 \;\Rightarrow\; |z| > 1/2$
2. $|z| > 1/2 \;\cap\; |z| < 1/4 \;\Rightarrow$ empty!
3. $|z| < 1/2 \;\cap\; |z| > 1/4 \;\Rightarrow\; 1/4 < |z| < 1/2$
4. $|z| < 1/2 \;\cap\; |z| < 1/4 \;\Rightarrow\; |z| < 1/4$

Step 4: Map each term back to the time domain.

Case 1. $x[n] = \frac{4}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{2}{3}\left(\frac{1}{4}\right)^n u[n]$

Case 2. no convergence!

Case 3. $x[n] = -\frac{4}{3}\left(\frac{1}{2}\right)^n u[-n-1] + 2/3\left(\frac{1}{4}\right)^n u[n]$

Case 3. $x[n] = -\frac{4}{3}\left(\frac{1}{2}\right)^n u[-n-1] - 2/3\left(\frac{1}{4}\right)^n u[-n-1]$

If the problem tells you $x[n]$ is right-sided then you know Case 1 is true.

Mar

Power series and inverse transforms

The partial fraction expansion we have seen so far is for z-transforms which can be expressed as polynomials. If $X(z)$ is itself just a polynomial we can just read off the $x[n]$:

$$X(z) = 4z^3 + 2z^2 + \frac{1}{4}z + 1 + 2z^{-1} + z^{-2}$$

$$x[n] = 4\delta[n+3] + 2\delta[n+2] + \frac{1}{4}\delta[n+1] + \delta[n] + 2\delta[n-1] + \delta[n-2]$$

Using calculus (namely Taylor Series) we can write non-polynomial $X(z)$ as an infinite polynomial

Ex

Example $X(z) = \ln(1+az^{-1})$ and $x[n]$ is right-sided

How do we get $\ln(1+az^{-1})$?

Start with $\ln(1+r)$: ← this is prereq stuff from calculus

$$\frac{d}{dr}\ln(1+r) = \frac{1}{1+r} = \sum_{n=0}^{\infty}(-r)^n$$

$$\ln(1+r) = \int \sum_{n=0}^{\infty}(-r)^n dr = \sum_{n=0}^{\infty} \frac{-1}{n+1}(-r)^{n+1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} r^n$$

plug in $r = az^{-1}$:

$$\ln(1+az^{-1}) = az^{-1} - \frac{1}{2}a^2z^{-2} + \frac{1}{3}a^3z^{-3} - \frac{1}{4}a^4z^{-4} + \dots$$

Then

$$x[n] = a\,\delta[n-1] - \frac{1}{2}a^2\delta[n-2] + \frac{1}{3}a^3\delta[n-3] - \dots$$

Main

z-transform properties

The z-transform has many of the same properties that our other transforms have. The major difference is that we have to make sure to figure out how the ROC changes.

!!!

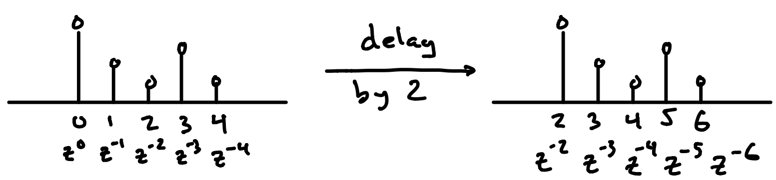
This is really crucial and is very easy to forget when you are doing z-transform manipulations. Be careful!

Linearity: $ax[n] + by[n] \xleftrightarrow{\mathcal{Z}} aX(z) + bY(z)$

$$\text{ROC} = \text{ROC}(x[n]) \cap \text{ROC}(y[n])$$

Time shifting: $x[n-n_0] \xleftrightarrow{\mathcal{Z}} z^{-n_0} X(z)$

$$\text{ROC} = \text{ROC}(x[n]) \pm \text{poles at } 0 \text{ or } \infty$$



Why do poles at 0 or ∞ change?

z^{-n_0} adds n_0 poles at 0 for $n > 0$

adds n_0 poles at ∞ for $n < 0$

Scaling: $z_0^n x[n] \xleftrightarrow{\mathcal{Z}} X(z/z_0)$

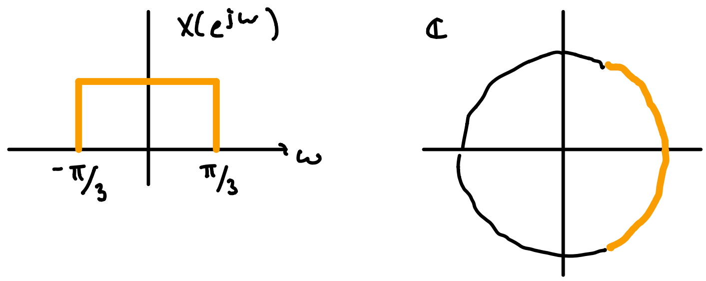
$$\text{ROC} = \text{ROC}(x[n]) \cdot |z_0|$$

What if $z_0 = e^{j\omega_0}$? Then

$$e^{j\omega_0 n} x[n] \xleftrightarrow{\mathcal{Z}} X(e^{-j\omega_0 n} z)$$

Remember that the DTFT of $x[n]$ corresponds to the unit circle $z = e^{j\omega}$.

We saw that $e^{j\omega_0 n} x[n]$ is a frequency shift for the DTFT. Since the DTFT is mapped to the unit circle in the z-transform:



A frequency shift corresponds to a <u>rotation</u> of the whole complex plane: $e^{j\omega_0} z \Rightarrow$ rotation by ω_0.

<u>Time Reversal</u>: $x[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X(\frac{1}{z})$

This makes a lot more sense if you understand the fact that $\alpha z^{-k} \longleftrightarrow x[k] = \alpha$

What about the ROC? Time reversal makes left-sided sequences right-sided and vice-versa, so

$$ROC = \frac{1}{ROC(x[n])}$$ so if the ROC is $a \le |z| \le b$, we get $\frac{1}{b} \le |z| \le \frac{1}{a}$.

<u>Upsampling</u>:

$$\underset{X(z)}{\overset{x[n]}{\longrightarrow}} \boxed{\uparrow K} \underset{Y(z) = X(z^k)}{\overset{y[n]}{\longrightarrow}}$$

$ROC = ROC(z^{1/k})$ ← k times slower, so the ROC scales accordingly.

This property also makes more sense if you understand the relationship between z^{-k} and $x[k]$.

!!! Getting a real understanding/sense of this mapping between the polynomial $X(z)$ and $x[n]$ makes z-transforms a lot easier. For some students it's

all very intuitive but for many it takes a little practice to see the pattern. As with many things, practice can help a lot.

Main **Conjugation:** $x^*[n] \xleftrightarrow{\mathcal{Z}} X^*(z^*)$

$ROC = ROC(x[n])$

Try Suppose $X(z)$ has a pole at b_1 and a zero at a_1 and corresponds to a signal $x[n]$. Where are the poles and zeros of $x^*[n]$?

"Differentiation": $n\,x[n] \xleftrightarrow{\mathcal{Z}} -z \frac{d}{dz} X(z)$

$$\frac{d}{dz} X(z) = \frac{d}{dz} \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} -n\, x[n] z^{-n-1}$$

$$= -z^{-1} \sum_{n=-\infty}^{\infty} n\, x[n] z^{-n}$$

$$= -z^{-1} \mathcal{Z}\{ n\, x[n] \}$$

and $ROC = ROC(x[n])$.

<u>Convolution</u>: The most important property — but by now it may be anticlimactic... sorry.

$$(x * h)[n] \xleftrightarrow{\mathcal{Z}} X(z)\,H(z)$$

What about the ROC? We need both x and h to converge, so $\text{ROC} = \text{ROC}(x[n]) \cap \text{ROC}(h[n])$

To see why this is important, consider our good old input-output LTI systems:

$$y[n-1] + y[n] = 2x[n] - 3x[n-1] + \tfrac{1}{2}x[n-2]$$

Take the z-transform

$$z^{-1}Y(z) + Y(z) = 2X(z) - 3z^{-1}X(z) + \tfrac{1}{2}z^{-2}X(z)$$

$$\Rightarrow \quad Y(z)(1+z^{-1}) = (2 - 3z^{-1} + \tfrac{1}{2}z^{-2})\,X(z)$$

or

$$Y(z) = \frac{2 - 3z^{-1} + \frac{1}{2}z^{-2}}{1-z^{-1}}\,X(z)$$

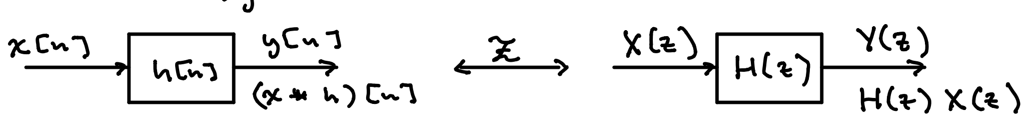
That means $H(z)$ for this system is

$$H(z) = \frac{2 - 3z^{-1} + \frac{1}{2}z^{-2}}{1-z^{-1}}$$

We can use partial fraction expansion to solve.

<u>System properties from the z-transform</u>

Since convolution in time is multiplication in the z-domain, we have the following picture for LTI systems:



We call the transform (CTFT, DTFT, Laplace, z) of the system impulse response the <u>transfer function</u> of the system:

$$H(z) = \frac{Y(z)}{X(z)}$$

Ex

<u>Example</u>: Suppose a LTI system has output

$$y[n] = \left(\tfrac{1}{3}\right)^n u[n]$$

to input signal $x[n] = \left(\frac{1}{2}\right)^n u[n]$. What is the impulse response of the system?

Find $X(z)$ and $Y(z)$:

$$X(z) = \mathcal{Z}\left\{ \left(\tfrac{1}{2}\right)^n u[n] \right\} = \frac{1}{1 - \frac{1}{2} z^{-1}}$$

$$Y(z) = \mathcal{Z}\left\{ \left(\tfrac{1}{3}\right)^n u[n] \right\} = \frac{1}{1 - \frac{1}{3} z^{-1}}$$

so

$$H(z) = \frac{1 - 1/3\, z^{-1}}{1 - 1/2\, z^{-1}} = \frac{1}{1 - \frac{1}{2} z^{-1}} - \frac{1/3\, z^{-1}}{1 - 1/2\, z^{-1}}$$

with

$$ROC = \underset{|z| > 1/2}{ROC(x)} \cap \underset{|z| > 1/3}{ROC(y)} = \left\{ z : |z| > \tfrac{1}{2} \right\}$$

So

$$h[n] = \left(\tfrac{1}{2}\right)^n u[n] - \tfrac{1}{3}\, \text{delay}_1 \left\{ \left(\tfrac{1}{2}\right)^n u[n] \right\}$$

$$= \left(\tfrac{1}{2}\right)^n u[n] - \tfrac{1}{3} \left(\tfrac{1}{2}\right)^{n-1} u[n-1]$$

We can check system properties (causality, stability, invertibility) from looking at $H(z)$.

<u>Causality</u>: we already saw how left- and right-sided signals relate to the z-transform.

Polynomial $H(z)$:

causal $\Rightarrow$ $H(z)$ has powers of $z \le 0$

ROC is the outside of a circle including ∞

anticausal $\Rightarrow$ $H(z)$ has powers of $z \geq 0$

ROC is the inside of a circle including 0

Rational $H(z)$: $H(z) = \dfrac{P(z)}{Q(z)}$

causal $\Rightarrow$ degree$(P(z)) \le$ degree$(Q(z))$

!!!

This is the degree in z, not in z^{-1}!

Ex

$$H(z) = \frac{1 - \frac{1}{2} z^{-1} + \frac{1}{3} z^{-2}}{1 - \frac{1}{6} z^{-1}}$$

$$= \frac{z^2 - \frac{1}{2} z + \frac{1}{3}}{z^2 - \frac{1}{3} z}$$

$\deg(P) = 2 = \deg(Q)$

CAUSAL

$$H(z) = \frac{\frac{1}{3} z^2}{1 + \frac{1}{2} z^{-1} + \frac{1}{4} z^{-3}}$$

$$= \frac{\frac{1}{3} z^5}{z^3 + \frac{1}{2} z^2 + \frac{1}{4}}$$

NOT CAUSAL

ROC is area outside largest magnitude pole.

Stability : $H(z)$ is stable if the ROC contains the unit circle $|z| = 1$

Why does this make sense? Thinking about the DTFT, if the ROC contains $|z| = 1$ then an input at any frequency passes through with a finite gain (doesn't blow up)

Invertibility: Algebraically, we can calculate an inverse system using the z-transform:

$$H^{-1}(z)\, H(z) = 1 \Rightarrow H^{-1}(z) = \frac{1}{H(z)}$$

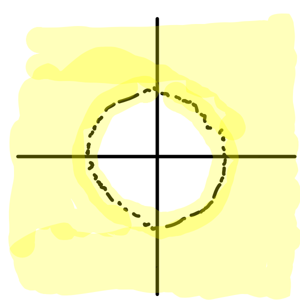
corresponds to $\delta[n]$

but we saw that the inverse system may not be causal and stable.

stable $\Rightarrow$ $H^{-1}(z)$ has ROC that contains the unit circle

causal $\Rightarrow$ $H^{-1}(z)$ has ROC that contains ∞

So the ROC has to contain range from $|z| = 1$ to ∞



$\Rightarrow$ no poles in this area

$\Rightarrow$ $H^{-1}(z)$ has als poles inside the unit circle

H(z) has a causal & stable inverse if and only if the poles of $H^{-1}(z)$ are all inside the unit circle.

Ex

Example: Suppose a system is defined by

$$y[n] - y[n-1] = x[n] - \frac{3}{4}x[n-1] + \frac{1}{8}x[n-2]$$

Does this system have a stable causal inverse?

Step 1: Take z-transforms

$$Y(z)\left(1 - z^{-1}\right) = X(z)\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 3/4\, z^{-1} + \frac{1}{8}z^{-2}}{1 - z^{-1}}$$

$$= \frac{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - 1/4\, z^{-1}\right)}{1 - z^{-1}}$$

Step 2: Invert & find poles & zeros

$$H^{-1}(z) = \frac{1 - z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}$$

zero at $z = 1$

poles at $z = \frac{1}{2}, \frac{1}{4}$

All poles inside the unit circle

$\Rightarrow$ has a causal, stable inverse

Example: what about the system defined by

$$y[n] - y[n-1] = x[n] + 2x[n-1] + \frac{3}{4}x[n-2]$$

$$Y(z)(1-z^{-1}) = X(z)\left(1+2z^{-1}+\frac{3}{4}z^{-2}\right)$$

$$= X(z)\left(1+\frac{3}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)$$

$$H(z) = \frac{\left(1+\frac{3}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)}{(1-z^{-1})}$$

$$H^{-1}(z) = \frac{1-z^{-1}}{\left(1+\frac{3}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)}$$

poles at $z = -\frac{3}{2}$, $-\frac{1}{2}$

zero at $z = 1$

pole @ $z = -3/2$ is <u>outside</u> the unit circle $\rightarrow$ no stable, causal inverse.