# Notes on Laplace Transforms

Objectives:

- be able to compute Laplace transforms using the definition
- explain the region of convergence (ROC) and how to find it
- use ROC properties to find the ROC for linear combinations of signals
- identify signal/system properties from the Laplace transform and ROC
- use Laplace transform pairs and properties to compute new Laplace transforms.

Main

# Laplace Transforms

A major theme in the study of LTI systems is the notion of a frequency-domain view of how systems operate. We have seen that complex exponentials are eigenfunctions of LTI systems:

$$\left( e^{st} * h(t) \right) = H(s)\, e^{st} \qquad s \in \mathbb{C}$$

$$H(s) \in \mathbb{C}$$

$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow H(s)\, e^{st}$$

That is, if we put a complex exponential into an LTI system, we get the same complex exponential at the output, scaled by a complex number $H(s)$ that depends on $s$. This scaling factor has a magnitude and phase:

$$H(s) = |H(s)|\, e^{j \measuredangle H(s)}$$

So we can see that $e^{st}$ gets a gain of $|H(s)|$ and a phase shift of $\measuredangle H(s)$.

This scaling factor $H(s)$ is a function of the system impulse response:

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$$

This is the bilateral Laplace transform of $h(t)$.

Def

**Def.** The <u>bilateral Laplace transform</u> of a CT signal $x(t)$ is

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

The <u>unilateral Laplace transform</u> of $x(t)$ is

$$X_u(s) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt$$

Not

Most textbooks discuss either the unilateral or bilateral Laplace transform in more detail and leave the other as a side note.

Mathematically, the difference is just in the limits of integration:

$$\text{bilateral} \longrightarrow \int_{-\infty}^{\infty}$$

$$\text{unilateral} \longrightarrow \int_{0}^{\infty}$$

So why do we need two transforms?!! This is confusing! It turns out they are useful for different things:

bilateral $\longrightarrow$ more general, understand noncausal systems, etc.

unilateral $\longrightarrow$ gives more insight into causal systems, can be used to solve diff eq.s with nonzero initial conditions, etc.

For now we will focus on bilateral Laplace transforms and will come back to the unilateral case later.

Main

What about the inverse transform? How can we go from $H(s)$ back to $h(t)$? This turns out to be a more complex problem (pun intended).

Let's do some examples first.

Ex

<u>Example</u>: Find the Laplace transform of

$$x(t) = e^{-at} u(t) \qquad a \in \mathbb{R} \quad a > 0$$

Just plug in and do the integral:

$$X(s) = \int_{-\infty}^{\infty} e^{-at} e^{-st} u(t)\, dt$$

$u(t)$ makes the signal causal, so unilateral and bilateral are the same!

$$= \int_{0}^{\infty} e^{-(s+a)t}\, dt$$

$$= \frac{1}{s+a}$$

But something is missing here: $s \in \mathbb{C}$ is a complex number. Let's write it as $s = \sigma + j\omega$:

$$X(s) = \frac{1}{(\sigma+a)+j\omega}$$

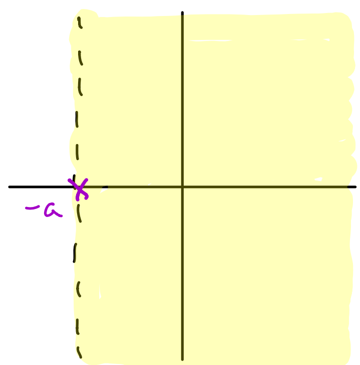
This only holds if $\sigma + a > 0$

So

$$X(s) = \frac{1}{s+j\omega} \quad \text{if} \quad \Re\{s\} > -a$$

Main

This example shows an important concept when dealing with Laplace Transforms (and other transforms): the formula for $X(s)$ in the $s$ domain only holds for certain values of $s$. In the example, $\mathrm{Re}\{s\} > -a$. If we visualize the complex plane:




this set $\{s : \mathrm{Re}\{s\} > -a\}$ is called the <u>Region of Convergence</u> (ROC) of $X(s)$

The ROC is very important in understanding the behavior and properties of LTI systems.

Not

We will use the following notation to describe Laplace transform pairs:

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \text{ ROC description}$$

So for example, we just showed in the example that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad \mathrm{Re}\{s\} > -a$$

We use a lower case $x$ for the time domain and upper case $X$ for the $s$-domain (or Laplace domain)

Main

As a special case of the example, we can set $a = 0$ to get $x(t) = u(t)$:

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}, \quad \mathrm{Re}\{s\} > 0$$

If we think back to our description of $u(t)$ as a system, this says that an integrator has a Laplace transform of $1/s$.

Ex

<u>Example</u>: Bilateral transforms are needed to understand non-causal systems. Suppose

$$x(t) = -e^{-at} u(-t)$$

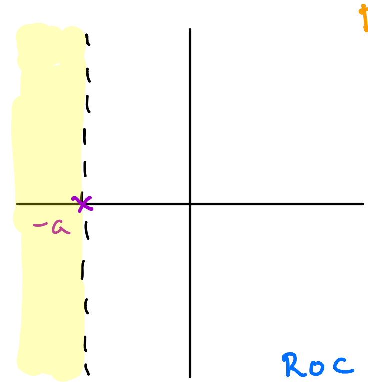
Then

$$X(s) = \int_{-\infty}^{\infty} -e^{-at} u(-t) e^{-st} dt$$

$$= -\int_{-\infty}^{0} e^{-(s+a)t} dt$$

$$= \frac{1}{s+a} \qquad \mathrm{Re}\{s\} < -a$$

This is the opposite from before!




ROC plot

Main

## Rational Transforms

One nice property of Laplace Transforms is that they are <u>linear</u>:

If $x_1 \xleftrightarrow{\mathcal{L}} X_1(s)$, $R_1$

$x_2 \xleftrightarrow{\mathcal{L}} X_2(s)$, $R_2$

then

$$a\,x_1(t) + b\,x_2(t) \xleftrightarrow{\mathcal{L}} a_1 X_1(s) + a_2 X_2(s)$$

ROC contains $R_1 \cap R_2$

<u>Note</u>: $R_1 \cap R_2$ might be empty! In this case the Laplace transform doesn't converge anywhere!

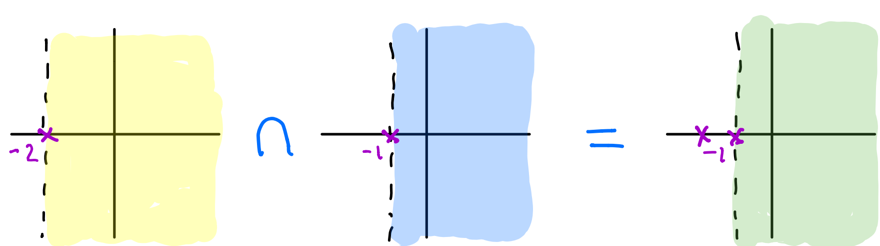
<u>Ex.</u> Suppose $x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$

From before,

$$3e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{3}{s+2} \qquad Re\{s\} > -2$$

$$2e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{-2}{s+1} \qquad Re\{s\} > -1$$

Intersecting the two ROC's:




We get

$$x(t) \xleftrightarrow{\mathcal{L}} \frac{s-1}{s^2+3s+2}, \quad Re\{s\} > -1$$

Main

So why do we say the ROC <u>contains</u> $R_1 \cap R_2$? It turns out we can have <u>pole-zero cancellation</u>.

Ex

<u>Example:</u> Suppose $x(t) = x_1(t) - x_2(t)$

where

$$X_1(s) = \frac{1}{s+1} \qquad Re\{s\} > -1$$

$$X_2(s) = \frac{1}{(s+1)(s+2)} \qquad Re\{s\} > -1$$

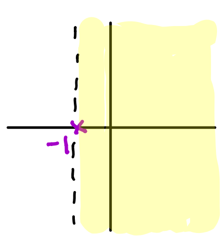
Then

$$X_1(s) - X_2(s) = \frac{s+2 \quad - \quad 1}{(s+1)\ (s+2)} = \frac{(s+1)}{(s+1)(s+2)}$$

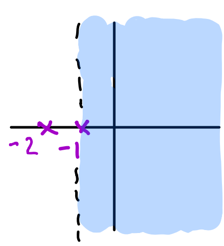
$$= \frac{1}{s+2}$$

The pole at $s=1$ was cancelled out by a zero!

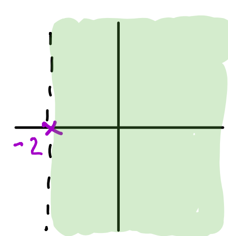
The ROC for this is $Re\{s\} > -2$, which contains $R_1 \cap R_2 = Re\{s\} > -1$.




$R_1$




$R_2$




$R_1 \cap R_2 \subset R$

Ex

**Ex.** Suppose $x(t) = e^{-2t}u(t) + e^{-t}(\cos(3t))u(t)$

What is its Laplace transform?

We have

$$e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+2} \qquad Re\{s\} > -2$$

but how do we deal with the other term?

Euler's relation!

$$e^{-t}\cos(3t) = \frac{1}{2}e^{-(1-3j)t} + \frac{1}{2}e^{-(1+3j)t}$$

$$\frac{1}{2}e^{-(1-3j)t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+(1-3j)} \qquad Re\{s\} > -1$$

$$\frac{1}{2}e^{-(1+3j)t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+(1+3j)} \qquad Re\{s\} > -1$$

Intersecting the <u>three</u> ROCs, we get

$$Re\{s\} > -1$$

$$X(s) = \frac{1}{s+2} + \frac{1}{2}\frac{1}{s+(1-3j)} + \frac{1}{2}\frac{1}{s+(1+3j)}$$

$$= \frac{2s^2+5s+12}{(s^2+2s+10)(s+2)} \qquad \text{(mmm, Algebra)}$$

Main

All of our examples so far have had the nice property that the Laplace transform is <u>rational</u> — it is the ratio of two polynomials:

$$X(s) = \frac{N(s)}{D(s)} \begin{array}{l} \leftarrow \text{numerator} \\ \leftarrow \text{denominator} \end{array}$$

This is happening because all of our $x(t)$ examples are linear combinations of complex exponentials.

Not We will write

LCCE for Linear Combination of Complex Exponentials

Main Since each complex exponential contributes a term like $\frac{1}{s+a}$,

an LCCE

$$\sum_{i=1}^{N} c_i e^{-a_i t} u(t)$$

becomes

$$\sum_{i=1}^{N} \frac{c_i}{s+a_i} \qquad \mathrm{Re}\{s\} > -a_i \quad i=1\cdots N$$

$$\Downarrow$$

$$\mathrm{Re}\{s\} > -\min_i \{a_i\}$$

When you combine the terms, you get a rational function.

There is a lot we can understand from rational Laplace transforms. One point of connection between rational transforms and things you may have seen before is through the solution of Linear Constant Coefficient Differential Equations (LCCDEs). We will return to that connection later.

Poles and Zeros: If $X(s) = \frac{N(s)}{D(s)}$ where $N(s)$ and $D(s)$ are polynomials in $s$, we can factorize them:

$$X(s) = C \frac{\prod_{i=1}^{N} (s - a_i)}{\prod_{j=1}^{M} (s - b_j)}$$

We call $\{a_i\}$ the zeros of $X(s)$ since $X(a_i) = 0$ and we call $\{b_j\}$ the poles of $X(s)$ since $|X(s)| \to \infty$ as $s \to b_j$. That is, the poles are where $X(s)$ "blows up." The ROC of $X(s)$ cannot contain any poles — the poles are points where $X(s)$ doesn't converge.

The diagram we saw before is a pole-zero plot of $X(s)$: we put an x for the poles and a o for the zeros.

Ex

Ex. For our previous example:

$$X(s) = \frac{2s^2 + 5s + 12}{(s^2 + 2s + 10)(s+2)}$$

$$s^2 + \frac{5}{2}s + 6 = 0$$

$$\frac{-5/2 \pm \sqrt{\frac{25}{4} - 24}}{2}$$

$$= -\frac{5}{4} \pm \frac{1}{4}\sqrt{71}\,j$$

$$= \frac{2s^2 + 5s + 12}{(s - (-1+3j))(s - (-1-3j))\ (s+2)}$$

$$= 2 \frac{\left(s - \left(-5/4 + \frac{1}{4}\sqrt{71}\,j\right)\right)\left(s - \left(-5/4 - 1/4\sqrt{71}\,j\right)\right)}{(s - (-1+3j))(s - (-1-3j))\ (s+2)}$$

So $a_1 = -5/4 + \frac{1}{4}\sqrt{71}\,j$

$a_2 = -5/4 - \frac{1}{4}\sqrt{71}\,j$

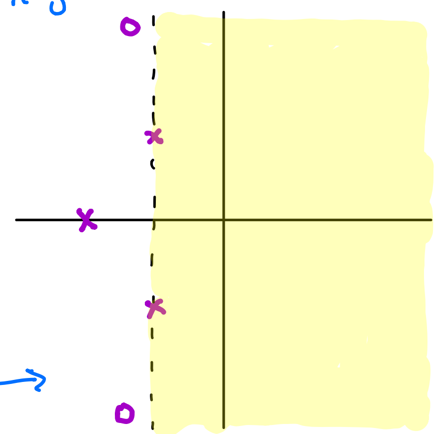
$b_1 = -1 + 3j$

$b_2 = -1 - 3j$

$b_3 = -2$

The pole-zero diagram $\longrightarrow$

(note, not to scale).

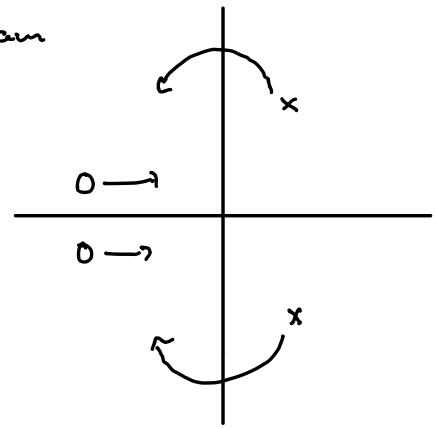


Main

As you can see, pole-zero diagrams can involve some messy calculations and the exact locations of the poles and zeros may be messy as well. A few things to keep in mind:

- In practice much filter design is done graphically — click to place the poles and zeros. The MATLAB filter design toolbox lets you do this.

- When designing control systems, the goal is often to use feedback to move the overall system poles and zeros into a more favorable position. This is the heart of the "root locus" method — it looks a bit like an (American) football play diagram



## The ROC and Signal / System Properties

We can tell a lot about a signal or system by looking at the Laplace transform:

1) The ROC of $X(s)$ consists of strips parallel to the imaginary axis.

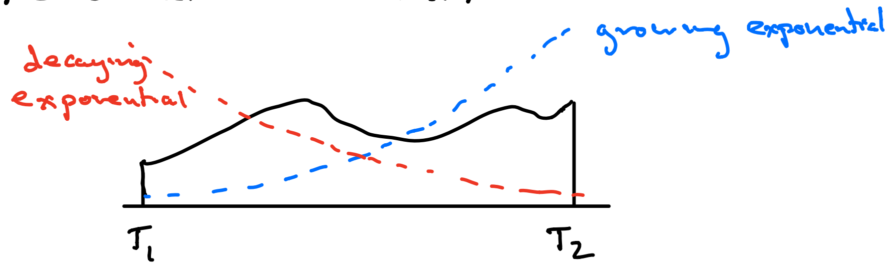
   → this is true of all of the examples we have seen so far.

2) For rational $X(s)$, the ROC contains no poles.

3) If $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire complex plane.

   This follows because the term $e^{st}$

cannot become arbitrarily large over the duration of $x(t)$:




So $\displaystyle\int_{T_1}^{T_2} |e^{-st} x(t)|\, dt < \infty$, i.e it converges

Note: absolute integrability was our condition for a system to be stable.

4) If $x(t)$ is right-sided ($=0$ for $t < T_0$) and if the line $\{s : Re\{s\} = \sigma_0\}$ is in the ROC, then

$$\{s : Re\{s\} > \sigma_0\}$$

is in the ROC.

Note: causal signals are right-sided

This says that if

$$\int_{T_0}^{\infty} e^{-\sigma_0 t} x(t)\, dt \quad \text{converges}$$

then $\displaystyle\int_{T_0}^{\infty} e^{-(\sigma_0 + \delta)t} x(t)$ converges.

5) If $x(t)$ is left-sided ($=0$ for $t > T_0$) and if the line $\{s : Re\{s\} = \sigma_0\}$ is in the ROC, then

$$\{s : Re\{s\} < \sigma_0\}$$

is in the ROC.

6) If $x(t)$ is two-sided and if the line $\{s : Re\{s\} = \sigma_0\}$ is in the ROC then the ROC consists of a strip in the s-plane that includes this line.

7) If $X(s)$ is rational then the ROC is bounded by poles or extends to infinity

- If $X(s)$ is also right-sided then the ROC is the region to the right of the rightmost pole.
- If $X(s)$ is also left-sided then the ROC is the region to the left of the leftmost pole.

!!! For a given Laplace transform $X(s)$ then, we might have many different possible ROCs and therefore several different time domain signals may have the same $X(s)$. That is why we need to specify both

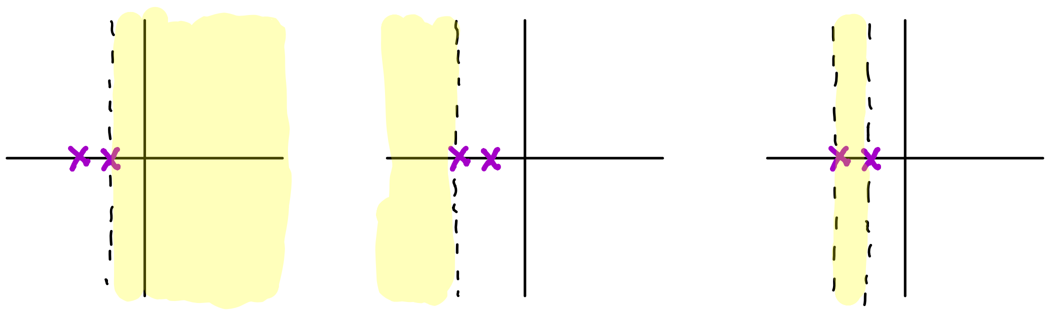
# X(s) and the ROC.

Ex

__Example__: Suppose $X(s) = \dfrac{1}{(s+1)(s+2)}$

Then we have poles at $s=-1$ and $s=-2$.

We have 3 choices for the ROC:



Right-sided Left-sided Two-sided

Ex

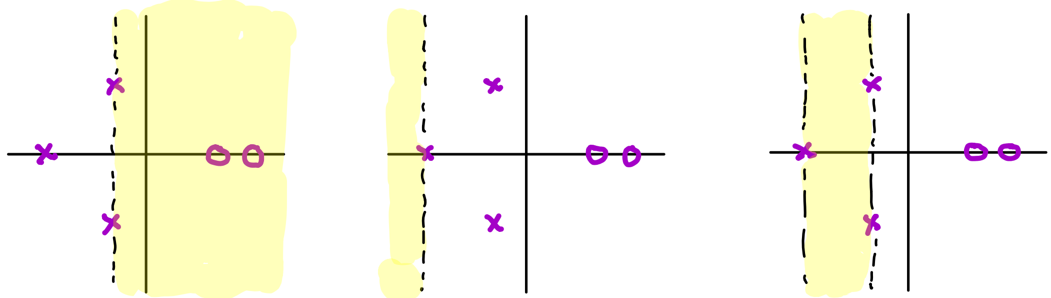
__Example__: Suppose

$$X(s) = \frac{s^2 - 5s + 6}{(s^2+2s+5)(s+3)}$$

Then $s^2+2s+5 \Rightarrow \dfrac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2j$

$$X(s) = \frac{(s-3)(s-2)}{(s-(-1-2j))(s-(-1+2j))(s+3)}$$

Now __remember__: the poles are what matter for the ROC, not the zeros.



Right-sided    Left-sided    Two-sided

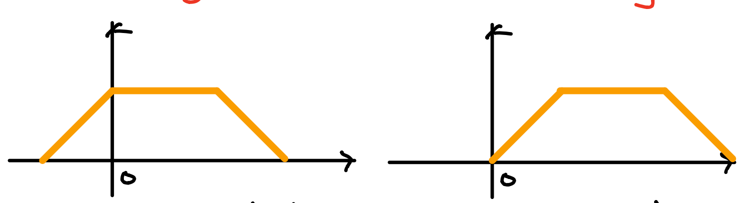
main

What about system properties? We have similar sorts of connections between the transform / ROC and system properties.

<u>Causality</u> : If $h(t)$ is causal then the ROC of $H(s)$ is a <u>right-half plane</u>, i.e. it has the form $\mathrm{Re}\{s\} > \alpha$.

!!!

The converse is <u>not true</u>. If the ROC is a right-half plane, all we can say is that $h(t)$ is right-sided. It may not be causal!




right-sided    causal

If $H(s)$ is rational then the ROC is the right half-plane to the right of the rightmost pole.

Ex

**Example:** $h(t) = e^{-|t|}$ not causal

But first find $H(s)$:

$$h(t) = e^{-t}u(t) + e^{t}u(-t)$$

$$H(s) = \underbrace{\frac{1}{s+1}}_{\mathrm{Re}\{s\}>-1} - \underbrace{\frac{1}{s-1}}_{\mathrm{Re}\{s\}<1} = \frac{-2}{s^2-1}$$

so the ROC for $H(s)$ is $-1 < \mathrm{Re}\{s\} < 1$

This is not a right-half plane.

Main

**Stability**: An LTI system is stable if and only if the ROC includes the imaginary (j) axis $\{s : \mathrm{Re}\{s\} = 0\}$.

Phil

This shows how important the transform evaluated at $s = j\omega$ is. In this case we get

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

This turns out to be the CT Fourier Transform (CTFT) of $h(t)$.

Ex

**Example:** Suppose $H(s) = \dfrac{s-1}{(s+1)(s-2)}$

There are 3 possible ROCs.

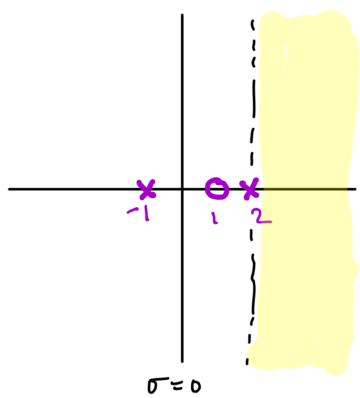
1) If the system is causal then the ROC is

$$\mathrm{Re}\{s\} > 2$$

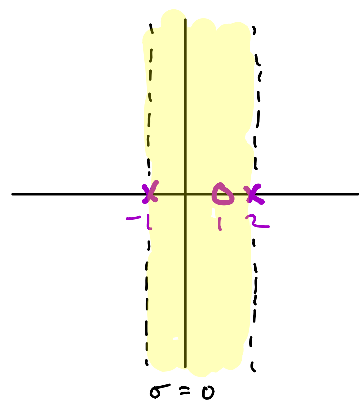
But this is unstable.

2) If the system is stable then the ROC has to be

$$-1 < \mathrm{Re}\{s\} < 2$$




causal + unstable




noncausal + stable

Main

Putting this together, we see that a causal system with rational $H(s)$ is stable if and only if all of the poles of $H(s)$ are in the left half of the s-plane. That is

$$\mathrm{Re}\{b_k\} < 0$$

for all poles.

Invertibility: We have seen before that while an inverse system may exist, it may not be stable or causal. Sometimes people call causal and stable systems <u>realizable</u>.

If $X(s)$ is rational, then finding an inverse system algebraically is "easy":

$$X(s) = \frac{N(s)}{D(s)} \Rightarrow X_i(s) = \frac{D(s)}{N(s)}$$

or

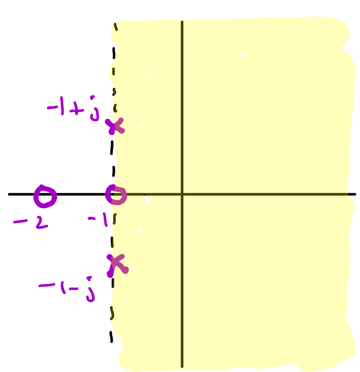
$$X_i(s) = P_N \frac{\prod_{j=1}^{N} (s - b_j)}{\prod_{i=1}^{M} (s - a_i)}$$

The poles of $X(s)$ become zeros of $X_i(s)$ and the zeros of $X(s)$ become poles of $X_i(s)$.

If this pole-zero diagram can have a stable and causal time-domain signal $x_i(t)$ then the system has a causal & stable inverse.

Ex

Example: $H(s) = \dfrac{s^2 + 3s + 2}{s^2 + 2s + 2}$ and $H(s)$ is causal

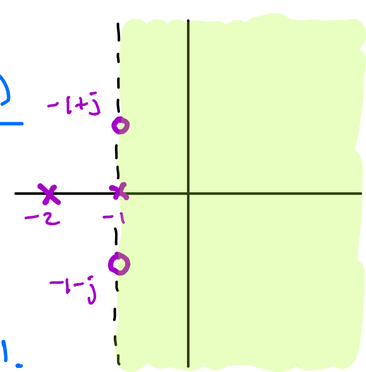



$$= \frac{(s+1)(s+2)}{(s+1+j)(s+1-j)}$$

$Re\{s\} > -1$

Then $H_i(s)$

$$= \frac{(s+1+j)(s+1-j)}{(s+1)(s+2)}$$




We can find an ROC for $H_i(s)$ so that $h_i(t)$ is stable and causal.

Try

Use partial fraction expansion on $H_i(s)$ above and find $h_i(t)$.

Try the example with $\frac{(s+1)(s-2)}{s^2+2s+2}$. Can you find a stable and causal inverse?

Main

## Inverting Laplace Transforms

The last two examples show that a given $X(s)$ might correspond to several different $x(t)$ signals. So how can we figure out what these $x(t)$ are?

For rational $X(s)$ we can use partial-fraction expansion to break apart the transform into a sum of terms. For example, if all of the poles are simple (first-order) then

$$X(s) = K + \sum_{k=1}^{N} \frac{c_k}{s-b_k}$$

and given the ROC, we can map each term in the sum to $e^{b_k t}u(t)$ or $-e^{b_k t}u(-t)$. The ROC tells us which terms should have right-sided and which should have left-sided transforms.

The easier case is where the numerator has <u>smaller</u> degree than the denominator. In this case,

$$K = \lim_{s \to \infty} X(s) = 0$$

and
$$X(s) = \sum_{k=1}^{N} \frac{c_k}{s - b_k}$$

For each $k = 1 \cdots N$,

$$\frac{c_k}{s - b_k} \overset{\mathcal{L}}{\longleftrightarrow} c_k e^{b_k t} u(t)$$

if ROC is to the right of $b_k$

$$\frac{c_k}{s - b_k} \overset{\mathcal{L}}{\longleftrightarrow} -c_k e^{b_k t} u(-t)$$

if ROC is to the left of $b_k$

Then add the terms up.

Ex

<u>Example:</u> $X(s) = \dfrac{1}{(s+1)(s+2)}$ $\quad \mathrm{Re}\{s\} > -1$

First, partial-fraction expansion:

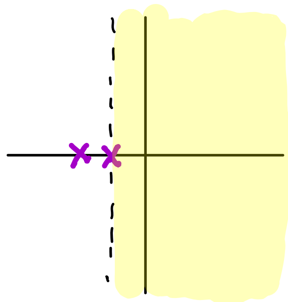
$$X(s)(s+1)\Big|_{s=-1} = 1$$

$$X(s)(s+2)\Big|_{s=-2} = -1$$

So

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

Now look at the ROC:

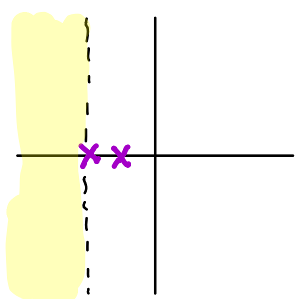


Right-sided

The ROC is to the right of both poles, so

$$x(t) = e^{-t}u(t) - e^{-2t}u(t)$$

What if the ROC was $\mathrm{Re}\{s\} < -2$ ?

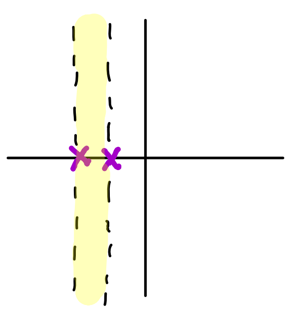


Left-sided

The ROC is to the left of both poles, so

$$x(t) = -e^{-t}u(-t) + e^{-2t}u(-t)$$

For the last case, the ROC could be $-2 < \mathrm{Re}\{s\} < -1$



Two-sided

The ROC is to the left of $-1$ and to the right of $-2$, so

$$x(t) = -e^{-t}u(-t) - e^{-2t}u(t)$$

Main

More generally, we can invert a Laplace transform by integrating it over a vertical line $\{s : \Re\{s\} = \sigma\}$ that is contained in the ROC.

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\, e^{st}\, ds$$

This requires doing line integrals in $\mathbb{C}$ which is a little too advanced for this class. We will generally stick to $X(s)$ made up out of signals whose transforms we know, like rational $X(s)$.

## Some Useful Laplace Transforms

Rational functions are great but there is a whole exciting world of Laplace transforms out there. You can find many Laplace transform tables on the internet, so the presentation here is not exhaustive.

The impulse function $\delta(t)$ is a bit weird, so let's see what happens when we transform it:

$$\int_{-\infty}^{\infty} \delta(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-s \cdot 0}\, dt = 1$$

What about the ROC? This holds for all $s$, so

$$\delta(t) \xleftrightarrow{\mathcal{L}} 1 \qquad \text{all } s \in \mathbb{C}$$

What about a delay, $\delta(t - t_0)$?

$$\int_{-\infty}^{\infty} \delta(t - t_0)\, e^{-st}\, dt = e^{-st_0} \qquad \text{for all } s \in \mathbb{C}$$

So

$$\delta(t - t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0} \qquad \text{for all } s \in \mathbb{C}$$

<u>Integrators and differentiators</u>

We already saw

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \qquad \mathrm{Re}\{s\} > 0$$

$$-u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \qquad \mathrm{Re}\{s\} < 0$$

We will show later that

$$x(t) * h(t) \xleftrightarrow{\mathcal{L}} X(s)\, H(s)$$

This is super-important!

So if we interpret $u(t)$ as an integrator:

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Then if we integrate multiple times

$$x(t) * \underbrace{u(t) * u(t) \cdots * u(t)}_{n \text{ times}}$$

Then we get $\frac{1}{s^n}$. But what is this?

$$
\begin{array}{ll}
n=1 & u(t) \\
n=2 & r(t) = t\,u(t) \\
n=3 & \frac{1}{2}t^2 u(t) \\
n=4 & \frac{1}{6}t^3 u(t) \\
\vdots & \Rightarrow \frac{1}{(n-1)!}t^{n-1}u(t)
\end{array}
$$

So

$$\frac{t^{n-1}}{(n-1)!}u(t) \longleftrightarrow \frac{1}{s^n} \quad \mathrm{Re}\{s\} > 0$$

$$-\frac{t^{n-1}}{(n-1)!}u(-t) \longleftrightarrow \frac{1}{s^n} \quad \mathrm{Re}\{s\} < 0$$

These are integrators. What about differentiation?

Well,

$$r(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s^2}$$

$$\frac{d}{dt}r(t) = u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}$$

So it seems that to get $X(s) = s$ we need

$$\frac{d}{dt}\delta(t) \xleftrightarrow{\mathcal{L}} s \quad \text{for all } s \in \mathbb{C}$$

Extrapolating from here,

$$\frac{d^n}{(dt)^n}\delta(t) \xleftrightarrow{\mathcal{L}} s^n \quad \text{for all } s \in \mathbb{C}$$

!!! Taking the derivative of a delta function is pretty weird, but this is an instance where explaining in detail might be too complicated.

Nota: We will write $X(s) = \mathcal{L}\{x(t)\}$ for Laplace transforms when it is more convenient to do so.

Phil: Integrators and differentiators are the "bread and butter" of analog control using circuits. Together with a simple gain, control systems built with these systems are called

Proportional - Integral - Derivative

or PID controllers.

## Sinusoids and Complex exponentials

We've seen a few of these:

$$e^{-at} u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \qquad \mathrm{Re}\{s\} > -a$$

$$-e^{-at} u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \qquad \mathrm{Re}\{s\} < -a$$

Taking integrals:

$$\frac{t^{n-1}}{(n-1)!} e^{-at} u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+a)^n} \qquad \mathrm{Re}\{s\} > -a$$

$$-\frac{t^{n-1}}{(n-1)!} e^{-at} u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+a)^n} \qquad \mathrm{Re}\{s\} < -a$$

What about sines and cosines? They are just made up out of complex exponentials:

$$\cos(\omega_o t)\,u(t) = \left(\frac{1}{2} e^{j\omega_o t} + \frac{1}{2} e^{-j\omega_o t}\right) u(t)$$

$$\mathcal{L}\{\cos(\omega_o t)u(t)\} = \frac{1}{2}\frac{1}{s-j\omega_o} + \frac{1}{2}\frac{1}{s+j\omega_o}$$

$$= \frac{s}{s^2+\omega_o^2} \qquad Re\{s\} > 0$$

$$\sin(\omega_o t)\,u(t) = \left(\frac{1}{2j} e^{j\omega_o t} - \frac{1}{2j} e^{-j\omega_o t}\right)$$

$$\mathcal{L}\{\sin(\omega_o t)u(t)\} = \frac{1/2j}{s-j\omega_o} - \frac{1/2j}{s+j\omega_o}$$

$$= \frac{\omega_o}{s^2+\omega_o^2} \qquad Re\{s\} > 0$$

So

$$\cos(\omega_o t)\,u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2+\omega_o^2} \qquad Re\{s\} > 0$$

$$\sin(\omega_o t)\,u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{\omega_o}{s^2+\omega_o^2} \qquad Re\{s\} > 0$$

Ex

Example: what about $e^{-at}\cos(\omega_o t)\,u(t)$

Using Euler's relation:

$$e^{-at}\cos(\omega_o t) = e^{-at}\left(\frac{1}{2} e^{j\omega_o t} + \frac{1}{2} e^{-j\omega_o t}\right)$$

So

$$\mathcal{L}\{e^{-at}\cos(\omega_o t)\} = \frac{1/2}{s+(\alpha-j\omega_o)} + \frac{1/2}{s+(\alpha+j\omega_o)}$$

$$= \frac{s+\alpha}{(s+\alpha)^2+\omega_o^2} \qquad Re\{s\} > -a$$

Try

Try the same process to $e^{-at}\sin(\omega_o t)\,u(t)$.

Main

## Laplace Transform Properties

Armed with some basic transforms, we can find the Laplace transform of lots of signals. To do this we need to understand the nice properties Laplace transforms have.

For all of these we will assume $\mathcal{L}\{x(t)\} = X(s)$ with ROC $R$, or for multiple signals

$$\underset{\text{with ROC } R_1}{\mathcal{L}\{x_1(t)\} = X_1(s)}, \quad \underset{\text{with ROC } R_2}{\mathcal{L}\{x_2(t)\} = X_2(s)}, \text{ etc.}$$

**Linearity:**

$$a x_1(t) + b x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} a X_1(s) + b X_2(s)$$

ROC contains $R_1 \cap R_2$

We saw this one before.

**Time shift:**

$$x(t - t_0) \overset{\mathcal{L}}{\longleftrightarrow} e^{-st_0} X(s)$$

with ROC $R$

Try

To show this, plug $x(t-t_0)$ into the definition and do a change of variables.

Main

**S-Domain Shift** $\quad e^{s_0 t} x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s - s_0)$

$\text{ROC} = R + \text{Re}\{s_0\}$

shift by $s_0$!

So if $s_0 > 0$ then we shift the ROC to the right by $s_0$

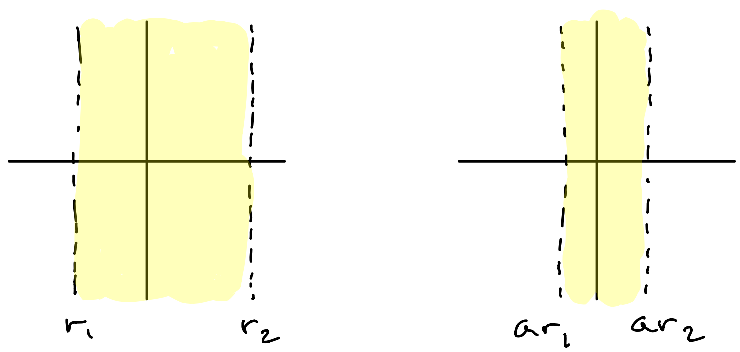
We can recover the example from above with this property. To show it, plug into the definition.

<u>Time scaling</u>

$$x(at) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

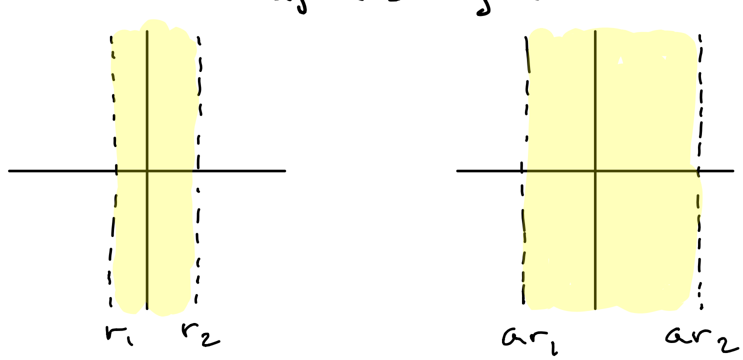
$$\text{ROC} = aR$$

So if $|a| < 1$ then we are stretching $x(t)$ out to get $x(at)$. The ROC shrinks by a factor of $a$ too:




Example for $a = 1/2$

If $|a| > 1$ then we are squishing $x(t)$ and the ROC expands by a factor of $a$.




Example for $a = 2$

<u>Conjugation:</u> $x^*(t) \xleftrightarrow{\mathcal{L}} X^*(s^*)$ ROC $= R$

note $s$ & $X$ are conjugated

This has a very useful implication: if $x(t)$ is real, then

$$x(t) = x^*(t)$$

$$X(s) = X^*(s^*)$$

so if $X(s)$ has a pole/zero at $s = s_0$ then it <u>also</u> has a pole/zero at $s = s_0^*$. That is, the poles and zeros appear in <u>conjugate pairs</u>.

<u>Convolution</u>

$$\boxed{x_1(t) * x_2(t) \xleftrightarrow{\mathcal{L}} X_1(s)\,X_2(s) \quad \text{ROC contains } R_1 \cap R_2}$$

This is one of the most important results on LTI systems:

"convolution in the time domain is multiplication in the transform domain"

Phil

We will see this fact in many forms — one for each transform in fact. The whole reason to study transforms is to make convolution easier to understand.

Main

## Differentiation in time

$$\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{L}} s\,X(s) \quad \text{ROC contains } R$$

We can show this by taking the derivative of the inverse transform:

$$\frac{d}{dt}x(t) = \frac{d}{dt}\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} e^{st}X(s)\,ds$$

pulling the derivative in

$$= \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} s\,e^{st}X(s)\,ds$$

## Differentiation in frequency

$$-t\,x(t) \xleftrightarrow{\mathcal{L}} \frac{d}{ds}X(s) \quad \text{ROC} = R$$

Use the same trick to show this using the transform (not the inverse transform)

Phil

You may notice some parallels between time & frequency transforms: $s\,X(s)$ and $-t\,x(t)$, etc. There is actually a nice theory of duality for many of these transforms but it's a bit out of scope for our class.

Ex

Example: What is the Laplace transform of

$$x(t) = t\, e^{-at} u(t) \quad ?$$

Start with

$$e^{-at} u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \quad \mathrm{Re}\{s\} > -a$$

$$t e^{-at} u(t) \xleftrightarrow{\mathcal{L}} -\frac{d}{ds}\left(\frac{1}{s+a}\right)$$

$$= \frac{1}{(s+a)^2}$$

Try

Try doing this one from the definition. You may need to do integration by parts.

See if you can find a general form for

$$\mathcal{L}\{t^n e^{-at} u(t)\}$$

Main

Integration in time:

$$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s} X(s) \quad \begin{array}{l}\text{ROC contains}\\ R \cap \{\mathrm{Re}\{s\} > 0\}\end{array}$$

Remember, the integrator is not stable, so the ROC cannot contain the imaginary axis.

Initial and Final Value Theorems

We can sometimes get more information about the time domain signal by looking at the Laplace transforms.

If $x(t) = 0$ for $t < 0$ (i.e. is <u>causal</u>)
and has no $\delta$ functions or other
singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} s\,X(s)$$ initial value theorem

approaching from the right

If $x(t)$ has a finite limit as $t \to \infty$
then

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} s\,X(s)$$ final value theorem

These are handy when working with rational transforms.