Unilateral Laplace Transforms

Learning Objectives

- Take unilateral Laplace transforms
- Use transform properties to simplify transforms
- Explain the connection between the LCCDE and LTI system analysis.
- Solve LCCDEs using Laplace transforms
- Find the zero-input / zero-state outputs of an LTI system based on an LCCDE with initial conditions
- Compute the transient and steady state behavior of systems governed by LCCDEs (or equivalently, with a rational transfer function).

Main

The unilateral Laplace transform is a special case of the bilateral Laplace transform restricted to $x(t)$ that are $=0$ for $t<0$.

Not !!!

Since causal LTI systems have an impulse response that is 0 for $t<0$, we sometimes call the impulse response causal as well. This is confusing — causality is a system property, not a signal property. And then to make things worse, we call signals causal if they are zero for $t<0$. What a terrible state of affairs! If you are learning this stuff for the first time then it can be tough. Here's a mental rewrite:

$x(t)$ is causal

"the LTI system with impulse response"

Not

So the unilateral Laplace transform is the same as the bilateral transform if $x(t)$ is causal.

$$X_u(s) = \int_0^\infty x(t)\, e^{-st}\, dt = \int_{-\infty}^\infty x(t)\, e^{-st}\, dt$$

if $x(t)$ causal

Another way to think about this is to take the bilateral Laplace transform of $x(t)\,u(t)$.

Why do we need the unilateral transform? It makes things simpler — since $x(t)$ is causal we know that the ROC of $X(s)$ is a right half-plane. That makes taking inverses for rational $X(s)$ very easy, since each term in the partial fraction expansion maps to the causal version. That means most of the ROC stuff that we had to care about for bilateral transforms goes away.

$$\boxed{\text{A signal } x(t) \text{ has a unique unilateral Laplace transform } X_u(s).}$$

We still have to care about the ROC to address issues of stability, since the ROC has to contain the imaginary axis.

We've already computed lots of Laplace transform pairs:

$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$\cos(\omega_0 t)u(t)$	$\frac{s}{s^2+\omega_0^2}$ etc...

Similarly, all of the properties from bilateral Laplace transforms carry over:

$x(t-T)u(t-T)$	$e^{-sT}X_i(s)$
$e^{at}x(t)$	$X(s+a)$
$x(at)$ $a>0$	$\frac{1}{a}X(\frac{s}{a})$ etc...

And of course, most importantly ...

$$\boxed{x_1(t) * x_2(t) \xleftrightarrow{\mathcal{L}} X_{i,1}(s)\, X_{i,2}(s)}$$

"convolution in the time domain is multiplication in the transform domain"

The unilateral Laplace transform is most useful for studying linear constant coefficient differential equations (LCCDEs). Lots of LTI systems can be expressed as LCCDEs — in particular, systems which have rational Laplace transforms.

To get a handle on this we need to look a bit more at the calculus-related Laplace transform properties:

$\frac{d}{dt} x(t)$	$s X(s) - x(0^-)$
$\frac{d}{dt^2} x(t)$	$s^2 X(s) - s\, x(0^-) - \frac{d}{dt} x(0^-)$
$\int_{0^-}^{t} x(\tau) d\tau$	$\frac{1}{s} X(s)$
$t\, x(t)$	$-\frac{d}{ds} X(s)$
$\frac{1}{t} x(t)$	$\int_{s}^{\infty} X(\sigma)\, d\sigma$
$x(0^+)$	$\lim_{s \to \infty} s X(s)$
$\lim_{t \to \infty} x(t)$	$\lim_{s \to 0} s X(s)$

Let's do some examples to get a feel for these.

Ex

Example. Find the Laplace transform of $\sin(\omega_0 t)u(t)$.

We have $\frac{d}{dt}\cos(\omega_0 t) = -\omega_0 \sin(\omega_0 t)$

thus: $-\frac{d}{dt}\frac{1}{\omega_0}\cos(\omega_0 t) \xleftrightarrow{\mathcal{L}} -\frac{s}{\omega_0}\left(\frac{s}{s^2+\omega_0^2} - \left(-\frac{1}{\omega_0}\right)\right)$

$$= \frac{-s^2/\omega_0}{s^2+\omega_0^2} + \frac{1}{\omega_0}$$

$$= \frac{\omega_0}{s^2+\omega_0^2}$$

But there are other ways we can do this:

$$\int_{0^-}^{t} \cos(\omega_0 \tau)\, d\tau = \frac{1}{\omega_0}\sin(\omega_0 t)$$

So

$$\sin(\omega_0 t)\, u(t) \xleftrightarrow{\mathcal{L}} \omega_0 \frac{1}{s} \frac{s}{s^2+\omega_0^2}$$

$$= \frac{\omega_0}{s^2+\omega_0^2}$$

That was a lot easier! The moral of the story (or example) is that much like doing integrals, finding the right Laplace transform properties to use is a little bit of an art — you can get a feel for the "tricks" as you do more examples.

Example: Find the Laplace transform of $\sin^2(2t-4)\,u(t)$

What to do? We want to manipulate the expression so we can use Laplace transform properties.

We can try to build it up in stages. We need our good old double angle formula:

$$\cos(2A) = \cos^2(A) - \sin^2(A)$$
$$= 1 - 2\sin^2(A)$$

$$\sin^2(A) = \frac{1}{2}\left(1 - \cos(2A)\right)$$

Now setting $A = 2t - 4$:

$$\sin^2(2t-4)\,u(t) = \frac{1}{2}\left(1 - \cos(4t-8)\right)u(t)$$

Taking Laplace transforms of both sides:

$$\mathcal{L}\{\sin^2(2t-4)\,u(t)\} = \frac{1}{2}\mathcal{L}\{u(t)\}$$
$$-\frac{1}{2}\mathcal{L}\{\cos(2t-8)\,u(t)\}$$

So now we need to find the Laplace Transform of $\cos(4t-8)\,u(t)$. The next step is to <u>Eulerize</u>:

$$\int_0^\infty \cos(4t-8)\,u(t)\,e^{-st}\,dt$$
$$= \int_0^\infty \frac{1}{2}e^{j(4t-8)}\,e^{-st}\,dt$$
$$+ \int_0^\infty \frac{1}{2}e^{-j(4t-8)}\,e^{-st}\,dt$$
$$= \frac{1}{2}e^{-j8}\,\mathcal{L}\{e^{j4t}\} + \frac{1}{2}e^{j8}\,\mathcal{L}\{e^{-j4t}\}$$
$$= \frac{1}{2}e^{-j8}\,\frac{1}{s-4j} + \frac{1}{2}e^{j8}\,\frac{1}{s+4j}$$
$$= \frac{1}{2}\left(\frac{e^{-j8}(s+4j) + e^{j8}(s-4j)}{s^2+16}\right)$$
$$= \frac{s\cos(8) - 4\sin(8)}{s^2+16}$$

Putting it together:

$$\mathcal{L}\{\sin^2(2t-4)\} = \frac{1}{2}\left(\frac{1}{s} - \frac{s\cos(8) - 4\sin(8)}{s^2+16}\right)$$

Main

Let's look at phase shifts more generally:

$$\mathcal{L}\{\cos(\alpha t-\beta)\} = \int_0^\infty \cos(\alpha t-\beta)e^{-st}\,dt$$

$$= \int_0^\infty \tfrac{1}{2} e^{j(\alpha t-\beta)} e^{-st} dt + \tfrac{1}{2}\int_0^\infty \tfrac{1}{2} e^{-j(\alpha t-\beta)} e^{-st} dt$$

$$= \tfrac{1}{2} e^{-j\beta} \frac{1}{s-j\alpha} + \tfrac{1}{2} e^{j\beta}\frac{1}{s+j\alpha}$$

$$= \tfrac{1}{2}\left(\frac{e^{-j\beta}(s+j\alpha) + e^{j\beta}(s-j\alpha)}{s^2+\alpha^2}\right)$$

$$= \frac{1}{s^2+\alpha^2}\left(s\cdot\tfrac{1}{2}\left(e^{j\beta}+e^{-j\beta}\right) + \alpha\,\tfrac{1}{2}\left(e^{j(\beta-\pi/2)} + e^{-j(\beta-\pi/2)}\right)\right)$$

$$= \frac{s\cos(\beta) - \alpha\sin(\beta)}{s^2+\alpha^2}$$

As a sanity check:

If $\beta = 2\pi$ then we get $\frac{s}{s^2+\alpha^2} = \mathcal{L}\{\cos(\alpha t)\}$

If $\beta = \pi/2$ then we get $\frac{\alpha}{s^2+\alpha^2} = \mathcal{L}\{\sin(\alpha t)\}$

If $\alpha = 2\pi f_0$ and $\beta = \pi$ then we get

$$\frac{-s}{s^2+(2\pi f_0)^2} = \mathcal{L}\{-\cos(\alpha t)\}$$

Ex

Example: Find the Laplace transform of $t^2 e^{-2t}\sin[\frac{\pi}{6}t]u(t)$

Start with $\mathcal{L}\{\sin(\pi/6\, t)u(t)\} = \frac{\pi/6}{s^2+(\pi/6)^2}$

Now take two derivatives:

$$\mathcal{L}\{t^2\sin(\pi/6\, t)u(t)\} = \frac{d}{ds}\frac{\pi/6\cdot 2s}{(s^2+(\pi/6)^2)^2}$$

$$= \frac{(\pi/3)\,(s^2+(\pi/6)^2)^{\cancel{2}} - (\pi/3)\, s\; 2\,\cancel{(s^2+(\pi/6)^2)}\; 2s}{(s^2+(\pi/6)^2)^{\cancel{4}\,3}}$$

$$= \frac{-\pi s^2 + (\pi/3)(\pi/6)^2}{(s^2+(\pi/6)^2)^3}$$

Finally, e^{-2t} sends $s \longrightarrow s+2$

$$\mathcal{L}\{ t^2 e^{-2t} \sin(\pi/6\, t)\, u(t) \}$$

$$= \frac{-\pi (s+2)^2 + (\pi/6)^2\, \pi/3}{((s+2)^2 + (\pi/6)^2)^3}$$

On the one hand, you might think this is horribly ugly. On the other hand, this is way easier than doing the integral

$$\int_{-\infty}^{\infty} t^2 e^{-2t} \sin(\pi/6\, t)\, u(t)\, dt$$

There are other ways to do this: we could have Eulerized $\sin(\pi/6\, t)$ and then $\mathcal{L}\{\ \}$ the two complex exponentials and then taken 2 derivatives.

<u>Example:</u> For the previous example, find $x(0^-)$ and $x(\infty)$.

Here $x(\infty) = \lim_{t\to\infty} x(t)$.

We use the initial and final value theorems:

$x(0^+) = \lim_{s\to\infty} sX(s) = 0$ since the denominator has s^6

$x(\infty) = \lim_{s\to 0} sX(s) = 0$ since the s doesn't cancel

These are very handy for studying transient and long term (steady-state) responses of systems.

Main

So what about differential equations? When we have a system as an LCCDE we can often interpret the input-output relationship in terms of derivatives:

$$q_N \left(\frac{d}{dt}\right)^N y(t) + q_{N-1}\left(\frac{d}{dt}\right)^{N-1} + \cdots + q_1 \frac{d}{dt} y(t) + q_0$$

$$= p_M \left(\frac{d}{dt}\right)^M x(t) + p_{M-1}\left(\frac{d}{dt}\right)^{M-1} x(t) + \cdots + p_1 \frac{d}{dt} x(t) + p_0$$

Taking Laplace transforms on both sides and assuming zero initial conditions, each derivative turns into an s:

$$\left(q_N s^N + q_{N-1} s^{N-1} + \cdots + q_1 s + q_0\right) Y(s)$$

$$= \left(p_M s^M + p_{M-1} s^{M-1} + \cdots + p_1 s + p_0\right) X(s)$$

Now, this input-output relationship is defining a system. This system is linear and time invariant.

Try

Try showing the system defined by the LCCDE above is linear and time invariant using the definitions of linearity and time invariance.

Main

LTI systems are characterized by their impulse response. Let's call the impulse response $h(t)$. How can we find it from the LCCDE? Remember, the output $y(t)$ is the convolution

$$y(t) = h(t) * x(t)$$

Convolution in time is multiplication in s:

$$Y(s) = H(s)X(s)$$

So we can write $H(s)$ as $H(s) = \dfrac{Y(s)}{X(s)}$. Thus is sometimes called the transfer function of the LTI system, since $H(s)$ changes $X(s)$ to $Y(s)$ (by multiplication).

For our LCCDE, $H(s)$ is rational:

$$H(s) = \frac{p_M s^M + p_{M-1} s^{M-1} + \cdots + p_1 s + p_0}{q_N s^N + q_{N-1} s^{N-1} + \cdots + q_1 s + q_0}$$

$$= C \, \frac{(s-a_M)(s-a_{M-1}) \cdots (s-a_1)}{(s-b_N)(s-b_{N-1}) \cdots (s-b_1)}$$

So as promised, LCCDES correspond to rational $H(s)$.

The transfer function view of systems makes formally inverting rational $H(s)$ eas:y

$$\frac{1}{H(s)} = \frac{1}{C} \cdot \frac{(s-b_N)(s-b_{N-1}) \cdots (s-b_1)}{(s-a_M)(s-a_{M-1}) \cdots (s-a_1)}$$

If $N > M$ then $1/H(s) = \frac{D(s)}{N(s)}$ has a higher degree in the numerator than the denominator — it is an improper rational function. That means as $s \to \infty$ there is a pole at ∞ so the inverse cannot be stable.

If $N \leq M$ then the poles/zeros of $H(s)$ become the zeros/poles of $1/H(s)$ so to make $1/H(s)$ stable we need all of the zeros of $H(s)$ to be in the left half plane.

Putting this together, a stable and causal system with rational transfer function $H(s)$ has a stable and causal inverse if and only if the # of poles & zeros is the same and all poles & zeros are in the LHP.

Ex

Example: Suppose $H(s) = \dfrac{4s+17}{s^2+7s+10}$.

Factorize:

$$H(s) = \frac{4s+17}{(s+5)(s+2)} \implies \begin{array}{l}\text{poles @ } -2, -5 \\ \text{zero @ } -17/4\end{array}$$

2 poles, 1 zero $\implies$ inverse is not causal & stable.

Example: Suppose $H(s) = \dfrac{s^2+3s+2}{s^2+s+3}$

poles & zeros is the same ✓✓

$$\frac{1}{H(s)} = \frac{s^2+s+3}{s^2+3s+2} = \frac{\left(s-\left(-\frac{1}{2}+\frac{1}{2}\sqrt{11}\right)\right)\left(s-\left(-\frac{1}{2}-\frac{1}{2}\sqrt{11}\right)\right)}{(s+2)(s+1)}$$

Partial fraction expansion:

$$\left.\frac{s^2+s+3}{(s+2)}\right|_{s=-1} = \frac{3}{1} = 3$$

$$\left.\frac{s^2+s+3}{(s+1)}\right|_{s=-2} = \frac{4-2+3}{-1} = -5$$

$$\frac{1}{H(s)} = \frac{3}{s+1} - \frac{5}{s+2}$$

$$h(t) = 3e^{-t}u(t) - 5e^{-2t}u(t)$$

take causal inverse
(unilateral Laplace transform)

Example:

Phil

The preceding fact is one of the key results in LTI system theory: since

$$\text{LCCDEs} \longleftrightarrow \text{rational } H(s)$$

this characterization of when stable/causal inverses exist covers all systems governed by simple differential equations.

The most common systems we have seen in earlier classes that correspond to LCCDEs come from RLC circuits and mass/spring systems — that is, from physics. When we study differential equations we learn to care about the initial conditions, the transient behavior, and the steady-state response. Our LTI system theory tells us that the response to an input $X(s)$ is $Y(s) = H(s)\,X(s)$. How do our diff. eq. ideas map onto the LTI system/convolution story?

We can think of initial conditions as if they were introducing an extra input term that is some multiple of $\delta(t)$, the impulse at 0. So the idea is to use linearity:

system with initial state A and input $x(t)$ $\longrightarrow$ system with zero initial state and input $x(t) + A\,\delta(t)$

So we can think of $Y(s)$ as having two components:

$$Y(s) = \underbrace{A\,H(s)\,1}_{\substack{\text{response from}\\ \text{initial state}\\ \text{"zero-input response"}}} + \underbrace{H(s)\,X(s)}_{\substack{\text{response due to}\\ \text{the input only}\\ \text{"zero state response"}}}$$

The standard transient + steady-state response is a different way of partitioning the output:

$$y(t) = \underbrace{y_{tr}(t)}_{\substack{\to 0 \text{ as}\\ t\to\infty}} + \underbrace{y_{ss}(t)}_{\substack{\not\to 0 \text{ as}\\ t\to\infty}}$$

Here we have to take the inverse Laplace transform and look at the terms to see which ones go to $y_{tr}(t)$ and which go to $y_{ss}(t)$.

Ex

Example. An LTI system is given by the following LCCDE:

$$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = \frac{d}{dt}x(t) + 4x(t)$$

Suppose the initial conditions are $y(0^-) = 2$, $\frac{d}{dt}y(0^-) = 0$. What all can we say about this system with input $e^{-t}u(t)$?

First find the transfer function.

$$s^2\,Y(s) + 5s\,Y(s) + 6 = s\,X(s) + 4\,X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+4}{s^2+5s+6} = \frac{s+4}{(s+3)(s+2)}$$

So two poles at $-2, -3$ and a zero at -4.

The zero-state response is the inverse Laplace transform of

$$H(s)\,X(s) = \frac{(s+4)}{(s+2)(s+3)} \cdot \frac{1}{(s+1)}$$ from $e^{-t}u(t)$

Doing a partial fraction expansion:

$$\left.\frac{s+4}{(s+2)(s+3)}\right|_{s=-1} = \frac{3}{1\cdot 2} = \frac{3}{2}$$

$$\left.\frac{s+4}{(s+1)(s+3)}\right|_{s=-2} = \frac{2}{-1\cdot 1} = -2$$

$$\left.\frac{s+4}{(s+1)(s+2)}\right|_{s=-3} = \frac{1}{-2\cdot -1} = \frac{1}{2}$$

So

$$Y_{zs}(s) = \frac{3/2}{s+1} - \frac{2}{s+2} + \frac{1/2}{s+3}$$

$$y_{zs}(t) = \frac{3}{2}e^{-t}u(t) - 2e^{-2t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

The zero-input response is the response to the initial conditions @ $t=0$. For this we need the Laplace transform with the initial conditions added in:

$$y(t) \longrightarrow Y(s)$$

$$\frac{d}{dt}y(t) \longrightarrow s\,Y(s) - y(0^-)$$

$$\frac{d^2}{dt^2}y(t) \longrightarrow s^2\,Y(s) - s\,y(0^-) - \frac{d}{dt}y(0^-)$$

So the LCCDE becomes

$$\left(s^2 Y(s) - s\,y(0^-) - \frac{d}{dt}y(0^-)\right) + 5\left(s\,Y(s) - y(0^-)\right) + 6\,Y(s)$$

$$= s\,X(s) - x(0^-) + 4\,X(s)$$

To get the zero-input response we set $x(0^-)$ to 0 and find the transfer function between the initial state and Y:

$$(s^2+5s+6)\,Y(s) = (s+5)\,s\,y(0^-) + \frac{d}{dt}y(0^-)$$

$$= 2(s+5)$$

So

$$Y_{ZI}(s) = \frac{2(s+5)}{(s+2)(s+3)}$$

Doing a partial fraction expansion:

$$\left.\frac{2(s+5)}{s+3}\right|_{s=-2} = \frac{6}{1} = 6$$

$$\left.\frac{2(s+5)}{s+2}\right|_{s=-3} = \frac{4}{-1} = -4$$

So

$$Y_{ZI}(s) = \frac{6}{s+2} - \frac{4}{s+3}$$

$$y_{ZI}(t) = 6e^{-2t}u(t) - 4e^{-3t}u(t)$$

And

$$y(t) = 6e^{-2t}u(t) - 4e^{-3t}u(t) + \tfrac{3}{2}e^{-t}u(t) - 2e^{-2t}u(t) + \tfrac{1}{2}e^{-3t}u(t)$$

All of these terms $\to 0$ so $y_{tr}(t) = y(t)$ and $y_{ss}(t) = 0$.