Notes 6: LTI Systems and complex exponentials

Objectives

- be able to show that the output of an LTI system with a complex exponential input is a scaled version of that complex exponential
- explain what the phrase "complex exponentials are eigenfunctions of LTI systems" means
- be able to set up the Laplace and Z-transforms to compute the scaling factors
- use algebra for complex functions to manipulate inputs into linear combinations of complex exponentials

Main

Complex exponentials through LTI Systems

Complex exponential functions

CT: e^{st} $\quad s \in \mathbb{C}$

DT: z^n $\quad z \in \mathbb{C}$

play a big role in LTI system analysis. That is because the output of an LTI system with a complex exponential is a scaled version of that exponential:

$$x(t) = e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = H(s)\, e^{st} \qquad H(s) \in \mathbb{C}$$

$$x[n] = z^n \longrightarrow \boxed{h(t)} \longrightarrow y[n] = H(z)\, z^n \qquad H(z) \in \mathbb{C}$$

!!!

An important thing to keep in mind/watch out for is that s and z are complex numbers, and so are $H(s)$ and $H(z)$. So when we say the output is the input scaled by a number, that scaling factor is complex, so it has a magnitude and phase:

$$e^{st} = e^{(Re\{s\} + j Im\{s\})t}$$

$$= e^{Re\{s\}t} \cdot e^{j\, Im\{s\}\, t} \quad \text{(mag/phase)}$$

$$H(s)e^{st} = |H(s)|\, e^{j \measuredangle H(s)}\, e^{st}$$

$$= \underbrace{|H(s)|\, e^{Re\{s\}t}}_{\text{magnitude}} \cdot \underbrace{e^{j(\measuredangle H(s) + Im\{s\}t)}}_{\text{phase}}$$

We are going to be working a lot with complex numbers and complex exponentials, so be careful!

Not

Linear systems and signals really involves a lot of linear algebra. We saw an example of that earlier: signals can be thought of as vectors and two signals are orthogonal if $\int_{-\infty}^{\infty} x(t)\, y(t)\, dt = 0$.

For complex exponentials we have

$$e^{st} * h(t) = H(s)\, e^{st}$$

$$z^n * h[n] = H(z)\, z^n$$

If signals act as vectors, LTI systems act like matrix multiplication. For a square $n \times n$ matrix A, if v is a vector such that

$$Av = \lambda v$$

then we call v an <u>eigenvector</u> of A and λ an <u>eigenvalue</u> of A.

Similarly we call complex exponentials <u>eigenfunctions</u> of LTI systems. $H(s)$ or $H(z)$ take the place of eigenvalues.

Def

<u>Def</u> An <u>eigenfunction</u> of an LTI system h is a signal x such that

$$(x * h)(t) = \lambda\, x(t)$$

or

$$(x * h)[n] = \lambda\, x[n]$$

We call λ an eigenvalue of the system.

Main

Theorem: Complex exponentials (CT or DT) are eigenfunctions of (CT or DT) LTI systems.

Proof. Let's do CT first. We just have to check the condition. Let h be an LTI system and $x(t) = e^{st}$:

$$(x * h)(t) = (h * x)(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau}_{=\lambda \text{ (if it converges)}}$$

!!! This should look familiar — it's the Laplace transform of $h(\tau)$.

For DT:

$$(x * h)[n] = (h * x)[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, z^{(n-k)}$$

$$= z^n \underbrace{\sum_{k=-\infty}^{\infty} h[k]\, z^{-k}}_{=\lambda \text{ if it converges}}$$

!!! This is called the z-transform of the impulse response

$\square$

So for LTI systems, the response to

e^{st} is $H(s)e^{st}$ where $H(s) = \int_{-\infty}^{\infty} e^{-s\tau} h(\tau) d\tau$

z^n is $H(z)\, z^n$ where $H(z) = \sum_{k=-\infty}^{\infty} z^{-k} h[k]$

From linearity of LTI systems,

$$\left(a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t} \right) * h(t)$$
$$= a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}$$

$$\left(a_1 z_1^n + a_2 z_2^n + a_3 z_3^n \right) * h[n]$$
$$= a_1 H(z_1) z_1^n + a_2 H(z_2) z_2^n + a_3 H(z_3) z_3^n$$

We can even extend this to infinite linear combinations

CT: $$\sum_{k=-\infty}^{\infty} a_k e^{s_k t} * h(t) = \sum_{k=-\infty}^{\infty} a_k H(s_k) e^{s_k t}$$

DT: $$\sum_{k=-\infty}^{\infty} a_k z_k^n * h[n] = \sum_{k=-\infty}^{\infty} a_k H(z_k) z_k^n$$

What this means is that if we know $H(s)$ and $H(z)$, we can easily write down the output of an LTI system for any signal that can be written as a linear combination of complex exponentials.

This raises an interesting question — what signals can we write as a linear combination of complex exponentials?

Ex

Examples: Suppose $y[n] = x[n-2]$ so

$$h[n] = \delta[n-2]$$

Then if

$$x[n] = e^{j4n}$$

we can find the output with

only $\neq 0$ at $k=2$

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]\, z^{-k}$$

$$= 1\, z^{-2}$$

Rewriting $x[n]$:

$$x[n] = (e^{j4})^n$$

We have $z = e^{j4}$ so

$$y[n] = H(e^{j4})\, e^{j4n}$$

$$= (e^{j4})^{-2}\, e^{j4n}$$

$$= e^{j(4n-8)}$$

If $x[n] = \cos(2\pi\, n/4)$, then we need to use our complex exponential identities:

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

So

$$x[n] = \frac{1}{2} e^{j2\pi n/4} + \frac{1}{2} e^{-j2\pi n/4}$$

This is the linear combination of two complex exponentials, so

$$y[n] = \tfrac{1}{2} H(e^{j2\pi/4}) e^{j2\pi n/4} + \tfrac{1}{2} H(e^{-j2\pi/4}) e^{-j2\pi n/4}$$

$$= \tfrac{1}{2} (e^{j\pi/2})^{-2} e^{j2\pi n/4} + \tfrac{1}{2} (e^{-j\pi/2})^{-2} e^{-j2\pi n/4}$$

$$= \tfrac{1}{2} e^{j(2\pi n/4 - \pi)} + \tfrac{1}{2} e^{-j(2\pi n/4 - \pi)}$$

$$= \cos(2\pi n/4 - \pi)$$

But you could have done this in a much simpler way: $h[n] = \delta[n-2]$ just delays the input by 2:

$$y[n] = x[n-2]$$

$$= \cos(2\pi (n-2)/4)$$

$$= \cos(2\pi n/4 - \pi)$$

$$= -\cos(2\pi n/4)$$

Not

This illustrates a few points:

- there are often multiple ways to solve a problem, and some are faster than others
- knowing what a system <u>does</u> can greatly ease computations
- turning cosines/sines into complex exponentials and back can be handy.

Main

The example shows that we can get a little better sense of what signals can be written as linear combinations of complex exponentials. This set also includes any signals that are linear combinations of sines and cosines.

Try

Try the previous example for a CT delay where $h(t) = \delta(t-2)$. Example 3.1 in the book may help.

Ex

<u>Example</u>: Suppose $h(t) = e^{j4\omega t} u(t)$

What is the response of $h(t)$ to an input

$$x(t) = 4\cos(t)\sin(2t) \qquad ?$$

① Draw a picture (in this case, use MATLAB)

② Rewrite $x(t)$ as a linear combination of complex exponentials:

$$x(t) = 4\left(\tfrac{1}{2}(e^{jt} + e^{-jt})\,\tfrac{1}{2}(e^{j2t} - e^{-j2t}\right)$$

$$= (e^{jt} + e^{-jt})(e^{j2t} - e^{-j2t})$$

$$= e^{j3t} + e^{jt} - e^{-jt} - e^{-j3t}$$

③ Find the Laplace (or z) transform:

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, u(t)\, dt$$

$$= \int_0^{\infty} e^{-(s+j4\omega t)}\, dt$$

$$= \left[\frac{-1}{s+j4\omega} e^{-(s+j4\omega)t} \right]_0^\infty$$

$$= \frac{1}{s+j4\omega}$$

④ Compute the outputs for each term:

$$y(t) = \frac{1}{j(3+4\omega)} e^{j3t} + \frac{1}{j(1+4\omega)} e^{jt} - \frac{1}{j(-1+4\omega)} e^{-jt} - \frac{1}{j(-3+4\omega)} e^{-j3t}$$

$$= \frac{1}{3+4\omega} e^{j(3t-\pi/4)} + \frac{1}{1+4\omega} e^{j(t-\pi/4)} + \frac{1}{-1+4\omega} e^{-j(t-\pi/4)} + \frac{1}{-3+4\omega} e^{-j(t-\pi/4)}$$

since $j = e^{j\pi/4}$