A note on invertibility for LTI systems

Main

I think the notion of invertibility is not very clearly explained in the book (or by me). So here is a standalone note that hopefully will be clearer. This definition is different than the one in the lecture notes / I gave in class.

Def

Def. An LTI system with impulse response $h(t)$ (or $h[n]$) has an <u>inverse system</u> with impulse response $h_i(t)$ (or $h_i[n]$) if:

$$(h * h_i)(t) = \delta(t) \qquad (CT)$$

or

$$(h * h_i)[n] = \delta[n] \qquad (DT)$$

The thing to note here is that this definition does not refer to any input or output signals. In the book, they (informally) say that h_i is an inverse system if the following two systems are equivalent:

$$x \longrightarrow \boxed{h} \longrightarrow \boxed{h_i} \longrightarrow x$$

$$x \longrightarrow \boxed{\delta} \longrightarrow x$$

From this they get the condition that

$$h * h_i = \delta$$

What is this definition missing?

First, if there are two signals x_1 and x_2 such that

$$x_1 * h = x_2 * h$$

then you should not be able to recover the input signal using any system h_i.

So this violates an intuitive sense of invertibility (the same one I used in class)

Ex

Example: Suppose $y[n] = x[n] + x[n-1]$.

Is this system invertible?

(1) $x[n] = y[n] - x[n-1]$ — rewriting as $x[n] =$ something

(2) $x[n-1] = y[n-1] - x[n-2]$ — delaying everything by 1

(3) $x[n] = y[n] - y[n-1] + x[n-2]$ — sub (2) into (1)

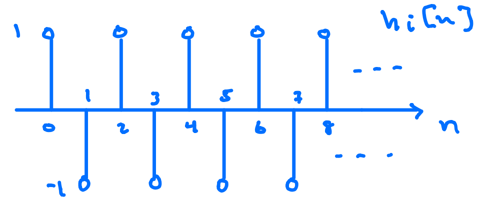
(4) $x[n-2] = y[n-2] - x[n-3]$ — delay (2) by 1

(5) $x[n] = y[n] - y[n-1] + y[n-2] - x[n-3]$ — sub (4) into (3)

$\vdots$

$$x[n] = \sum_{k=0}^{\infty} (-1)^k \, y[n-k]$$

$$h_i[n] = \sum_{k=0}^{\infty} (-1)^k \, \delta[n-k]$$



All this algebra works out, so what is wrong?

$h_i[n]$ is not <u>stable</u> : $\sum_{k=-\infty}^{\infty} |h[n]| = \infty$

Main

The example shows a problem with the definition: $h[n]$ may have an inverse $h_i[n]$ but that inverse is not stable.

That means the system h_i is not implementable.

In this case, the system is technically invertible (from the definition) but it fails to satisfy the condition that different inputs lead to different outputs:

$$x[n] = (-1)^n \Rightarrow y[n] = (-1)^n + (-1)^{n-1} = 0$$

$$x[n] = 3(-1)^n \Rightarrow y[n] = 3(-1)^n + 3(-1)^{n-1} = 0$$

But that's not the only problem with the definition!

Ex.

Example: $y[n] = x[n-1] - \frac{1}{2}x[n-2]$

Find the inverse system.

(1) $x[n-1] = y[n] + \frac{1}{2}x[n-2]$

(2) $x[n] = y[n+1] + \frac{1}{2}x[n-1]$

(3) $\quad\quad = y[n+1] + \frac{1}{2}y[n] + \frac{1}{4}x[n-2]$

(4) $x[n-2] = y[n-1] + \frac{1}{2}x[n-3]$

(5) $x[n] = y[n+1] + \frac{1}{2}y[n] + \frac{1}{4}y(n-1) + \frac{1}{8}x[n-3]$

$\vdots$

$$x[n] = \sum_{k=-1}^{\infty} \left(\frac{1}{2}\right)^{k+1} y[n-k]$$

$$h_i[n] = \sum_{k=-1}^{\infty} \left(\frac{1}{2}\right)^{k+1} y[n-k]$$

This system is stable:

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n = \frac{1}{1-1/2} = 2 < \infty$$

But there is another problem... h_i is not causal:

Main

This illustrates a second problem with our inverse definition: the inverse system may not be causal. This is important because for this block diagram to work,

$$x[n] \longrightarrow \boxed{h} \xrightarrow{y[n]} \boxed{h_i} \longrightarrow x[n]$$

to compute the output of h_i at time n we would need the output of h at a future time $n+1$. But that is not possible!

What can we do with this mess? From the definition, we can compute an inverse system, but that does not mean that

1) We can recover $x(t)$ or $x[n]$ from concatenating h and h_i.

2) The system h_i is implementable.

This seems like a pretty bad definition of invertibility. Some people define a system to be invertible if it has a stable, causal inverse. In our definition we say a system h_i is an <u>inverse</u> of h if $h * h_i = \delta$. This seems like splitting hairs, but that's the way things are.

So what should you do, as a student trying to a) learn this material for the first time and b) solve problems on homeworks and exams? The goal, first and foremost, is to demonstrate an understanding of the concepts.

1) For non-LTI systems, check if distinct inputs lead to distinct outputs.

- To show it is invertible, find an inverse system or prove that if $x_1 \neq x_2$ then $y_1 \neq y_2$

  <u>Ex.</u> $y(t) = x(t)^3$

  Then $x(t) = y(t)^{1/3}$

  Or: if $x_1 \neq x_2$ then there is a time t_0 such that

  $$x_1(t_0) \neq x_2(t_0)$$

  Then $y_1(t_0) = x_1(t_0)^3 \neq x_2(t_0) = y_2(t_0)$

  So the system is invertible

- To show it is <u>not</u> invertible, find a counterexample $x_1 \neq x_2$ such that $y_1 = y_2$

  <u>Ex.</u> $y[n] = x[n]\, x[n-1]$

  Let $x_1[n] = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

$$x_2[n] = \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Then $y_1[n] = y_2[n] = 0$.

2) For LTI systems, we should really ask you to find an <u>inverse system</u>. To do this you should find a way to express $x(t)$ (or $x[n]$) in terms of $y(t)$ (or $y[n]$)

<u>Ex.</u> $$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

By the Fundamental Theorem of Calculus, $x(t) = \frac{d}{dt} y(t)$

<u>Ex.</u> $$h[n] = \alpha^n u[n]$$

Then we have seen that

$$h_i[n] = \delta[n] - \alpha\,\delta[n-1]$$

is an inverse system:

$$\begin{aligned}(h * h_i)[n] &= \alpha^n u[n] - \alpha^{n-1} u[n-1] \\ &= \sum_{k=0}^{\infty} \alpha^k \delta[n-k] - \sum_{k=1}^{\infty} \alpha^k \delta[n-k] \\ &= \delta[n]\end{aligned}$$

<u>Ex.</u> $$y(t) = x(t) + \tfrac{1}{2} x(t-3)$$

We can use the same "shift and resubstitute" approach:

$$x(t) = y(t) - \tfrac{1}{2} x(t-3)$$

$$x(t-3) = y(t-3) - \tfrac{1}{2} x(t-6)$$

$$x(t) = y(t) - \tfrac{1}{2}\, y(t-3) + \tfrac{1}{4}\, x(t-6)$$

$$x(t-6) = y(t-6) - \tfrac{1}{2}\, x(t-9)$$

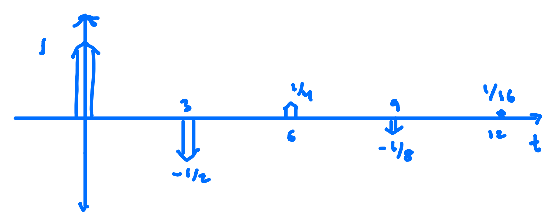
$$x(t) = y(t) - \tfrac{1}{2}\, y(t-3) + \tfrac{1}{4}\, y(t-6) - \tfrac{1}{8}\, x(t-9)$$

$$\vdots$$

$$x(t) = \sum_{k=0}^{\infty} \left(\tfrac{-1}{2}\right)^k y(t-3k)$$

So $$h_i(t) = \sum_{k=0}^{\infty} \left(-\tfrac{1}{2}\right)^k \delta(t-3k)$$

Is this causal?



impulse response only nonzero for $t \geq 0$

$\Rightarrow$ causal

Is this stable?

$$\int_{-\infty}^{\infty} |h_i(t)|\, dt = \int_{-\infty}^{\infty} \left| \sum_{k=0}^{\infty} \left(-\tfrac{1}{2}\right)^k \delta(t-3k) \right| dt$$

$$\leq \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^k \underbrace{\int_{-\infty}^{\infty} \delta(t-3k)\, dt}_{\text{1 impulse, area 1}}$$

$$= \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^k$$

$$= \frac{1}{1-1/2}$$

$$= 2$$

$$< \infty \quad \Rightarrow \text{stable}$$