Notes 5: Checking system properties for LTI Systems

Objectives

- Understand how the impulse response of an LTI system can be used to check if the system is
  - memoryless, causal, anticausal, or neither
  - invertible
  - stable
- compute the unit step response

Main

Just to recap: LTI systems (CT and DT) are completely described by their impulse response:

DT systems: $h[n]$ output when input is $\delta[n]$

CT systems: $h(t)$ output when input is $\delta(t)$

Earlier we saw that systems in general can have various properties:

- causality / memorylessness
- invertibility
- stability

This raises a natural questions for LTI systems:

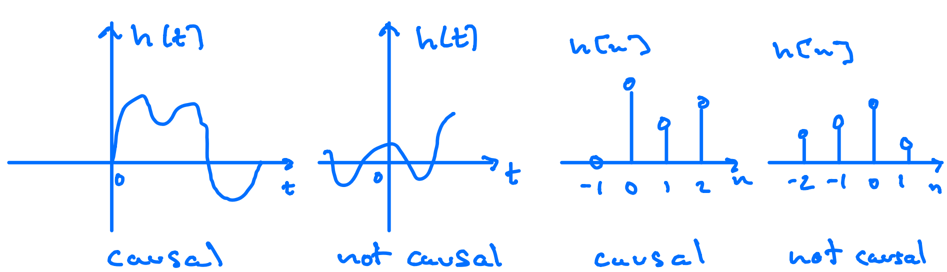
- how are these properties reflected in the impulse response?
- how can we check/verify system properties using the impulse response?

<u>Time issues</u> : causality and memorylessness

Checking for causality and memorylessness is easy if you can draw the impulse response:

Causal : output only depends on past & current inputs

$$\Rightarrow \begin{array}{ll} h(t) = 0 & \text{for } t < 0 \\ h[n] = 0 & \text{for } n < 0 \end{array}$$



Why is this true? Look at the convolution:

DT: Suppose that $h[n] = 0$ for $n < 0$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, \underbrace{h[n-k]}_{=0 \text{ for } k>n}$$

$$= \sum_{k=-\infty}^{n} x[k]$$

So $y[n]$ is only a function of the input x up to time n

CT: Suppose that $x(t) = 0$ for $t < 0$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, \underbrace{h(t-\tau)}_{=0 \text{ for } \tau>t}\, d\tau$$

$$= \int_{-\infty}^{t} x(\tau)\, d\tau$$

So $y(t)$ is only a function of the input x up to time t

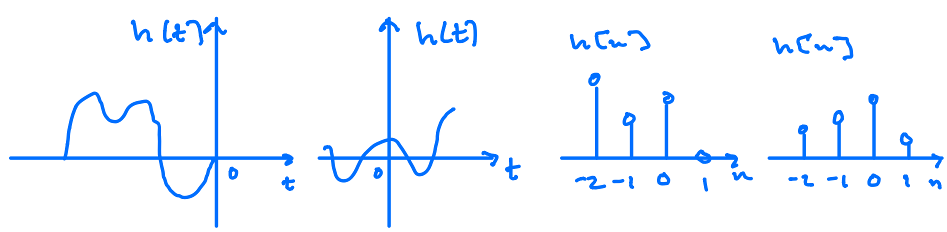
What about strictly causal? Only depends on past inputs:

$$\Rightarrow \begin{array}{l} h(t) = 0 \text{ for } t \le 0 \\ h[n] = 0 \text{ for } n \le 0 \end{array}$$

include 0

Anticausal: output depends only on current and future inputs:

$h(t) = 0$ for $t > 0$
$h[n] = 0$ for $n > 0$



anticausal | not anticausal | anticausal | not anticausal

Try

Try to show why this is true using the same type of argument as for the causal case.

Strictly anticausal:

$$h(t) = 0 \quad \text{for } t \geq 0$$
$$h[n] = 0 \quad \text{for } n \geq 0$$

Memoryless: output only depends on the current input

This is the easiest one:

$$h(t) = 0 \quad \text{for } t \neq 0$$
$$h[n] = 0 \quad \text{for } n \neq 0$$

This means memoryless systems are just scaled δ- functions:

$$h(t) = \alpha\, \delta(t)$$

$$h[n] = \beta\, \delta[n]$$

If $h(t)$ or $h[n]$ don't satisfy any of the previous conditions then the system is neither causal nor anticausal.

How do you check if a system is causal, anticausal, or memoryless?

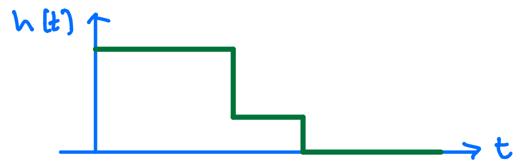
1. Draw a picture of the impulse response and check if it satisfies the conditions.

2. Work it out analytically to check where $h(t)$, $h[n] = 0$ (useful for step functions)

Ex

Example: Label each of these systems as memoryless, causal, anticausal, or neither causal nor anticausal.

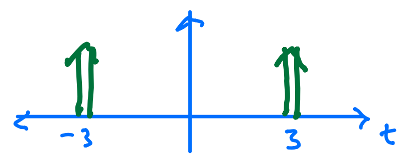
(a) $h(t) = 3u(t) - 2u(t-2) - u(t-3)$



$h(t) = 0$ for $t < 0$

$\Rightarrow$ causal

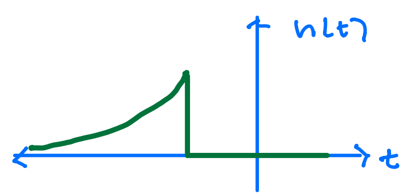
(b) $h(t) = \delta(t-3) + \delta(t+3)$



$h(-3) \neq 0$ and $h(3) \neq 0$

$\Rightarrow$ neither

(c) $h(t) = e^{t}\, u(-1-t)$



$h(t) = 0$ for $t > 0$

$\Rightarrow$ anticausal

(d) $h(t) = 4\,\delta(t)$

scaled $\delta(t)$ $\Rightarrow$ memoryless

Try

Make DT equivalents of the above examples and check their properties.

Ex.

Example: Label each of the following systems as memoryless, causal, anticausal, or neither.

(a) $y[n] - \alpha y[n-1] = x[n]$

① find $h[n]$: we did this before!

$$h[n] = \alpha^{n}\, u[n]$$

② Plot $h[n]$ or look at unit step properties:

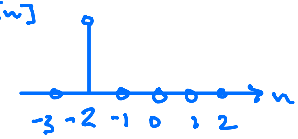
$$u[n] = 0 \quad \text{for} \quad n < 0$$

$\Rightarrow$ causal

(b) $y[n] = x[n+2]$

① $h[n] = \delta[n+2]$

② $h[n]$



$\Rightarrow$ anticausal

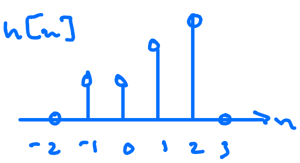
(c) $y[n] = 3\,x[n]$

Depends only on $x[n]$

$h[n] = 3\,\delta[n]$ $\Rightarrow$ memoryless

(d) $h[n] = 3\,\delta[n-2] + 2\,\delta[n-1] + \delta[n] + \delta[n+1]$

$h[n]$

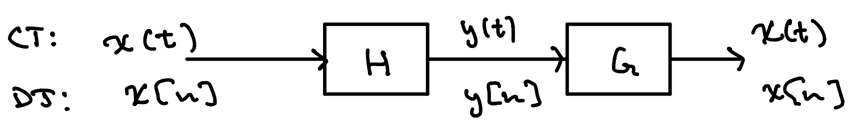


$h[n] \neq 0$ at some positive and negative times

$\Rightarrow$ neither

Main

Invertibility : a system H is invertible if there is another system G such that



We call G the inverse system

Questions:

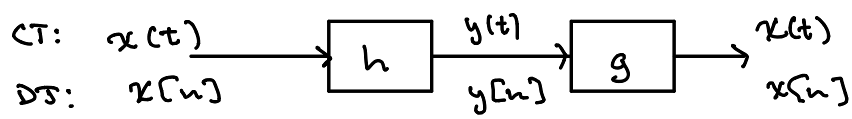
1) Is the inverse system of an LTI system also an LTI system?

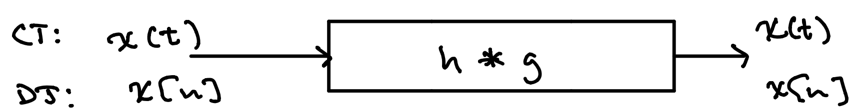
Yes. See problem 2.50.

2) How do we find the inverse system from the impulse response?

Sometimes this can be tricky... much easier with transform (Laplace, z) theory.

Note that for LTI systems, if h has inverse g then by associativity:





So $(h * g)(t) = \delta(t)$

or

$(h * g)[n] = \delta[n]$

So to find g we need to find a <u>system</u> that maps the <u>signal</u> $h[n]$ to $\delta[n]$.

Ex

<u>Example</u>: we saw some examples of this earlier:

Suppose $h[n] = \left(\frac{1}{3}\right)^n u[n]$

Then

$$h[n] = \delta[n] + \frac{1}{3}\delta[n-1] + \frac{1}{9}\delta[n-2] + \dots$$
$$= \delta[n] + \frac{1}{3} h[n-1]$$

So now we have a system with input

$h[n]$ and output $\delta[n]$:

$$\delta[n] = h[n] - \frac{1}{3} h[n-1]$$

The impulse response of this system is

$$g[n] = \delta[n] - \frac{1}{3}\delta[n-1].$$

<u>Example</u>: $h(t) = \delta(t - t_0)$ $\quad t_0 > 0$

Then $g(t) = \delta(t + t_0)$ — a delay by t_0 can be undone by advancing by t_0

<u>Note</u>: $h(t)$ is causal, $g(t)$ is anticausal.

<u>Stability</u>: A system is BIBO stable if for any bounded input: $|x[n]| \leq B$

$$|x(t)| \leq B$$

the output is also bounded: $|y[n]| \leq B'$

$$|y(t)| \leq B'$$

We can get a simple <u>sufficient</u> condition for stability:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \right|$$

$$\leq \sum_{k=-\infty}^{\infty} |x[k]\, h[n-k]|$$

$$= \sum_{k=-\infty}^{\infty} |x[k]| \cdot |h[n-k]|$$

If $|x[n]| \le B$

$$\le \sum_{k=-\infty}^{\infty} B \cdot |h[n-k]|$$

$$= B \cdot \sum_{k=-\infty}^{\infty} |h[k]|$$

So if $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$

then $h[n]$ is BIBO stable

Try

Show that if $\int_{-\infty}^{\infty} |h(\tau)| \, d\tau < \infty$

then $h(t)$ is BIBO stable using the same argument.

Note

Is the converse true? Yes! (see Prob 2.49)

That is, if a system is BIBO stable, then

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

or

$$\int_{-\infty}^{\infty} |h(t)| \, dt < \infty$$

Ex

Example. Suppose $h(t) = e^{-3t}u(t)$. Is $h(t)$ stable?

$$\int_{-\infty}^{\infty} |e^{-3t}u(t)|\, dt$$

$$= \int_{0}^{\infty} e^{-3t}\, dt$$

$$= 1/3 \qquad \text{Yes, stable.}$$

Example Suppose

$$h[n] = \sum_{k=1}^{\infty} \frac{1}{k} \delta[n-k]$$

$$= \delta[n-1] + \frac{1}{2}\delta[n-2] + \cdots$$

Then

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

harmonic series!

So this is not stable.

Main

This also brings up another important quantity: the unit step response of an LTI system:

$$s[n] = (u * h)[n] = \sum_{k=-\infty}^{n} x[k]$$

$$s(t) = (u * h)(t).$$

For DT systems,

$$h[n] = s[n] - s[n-1]$$

For CT systems,

$$h(t) = \frac{d}{dt} s(t)$$