Notes 4: Continuous-Time LTI Systems

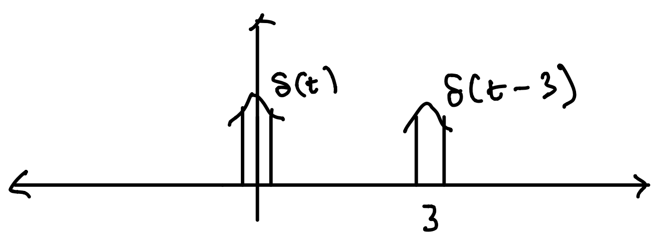
Objectives:

- Understand how the impulse response for CT LTI systems generates the output signal from the input signal
- Calculate system outputs using the convolution integral.
- Derive the 5 stages of the output signal for time-limited signals.
- use convolution properties to simplify calculations

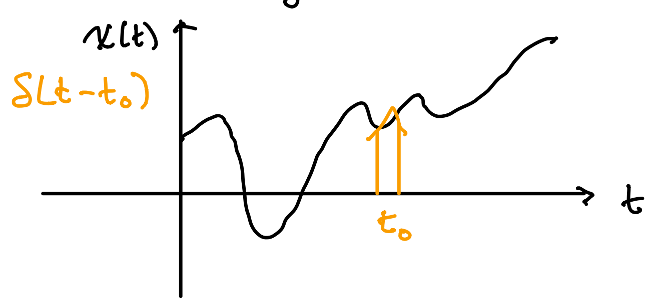
Main

Conceptually, CT systems don't differ that much from their DT counterparts. The mathematics is a bit different — sums become integrals, and so on.

The big mathematical difference is that we will have to really dig in on the properties of CT delta-functions. Remember that when integrated, $\delta(t)$ behaves as if there was an area of 1 exactly at $t=0$:



Mathematically:



So for any t_0...

$$\int_{-\infty}^{\infty} x(t)\, \delta(t-t_0)\, dt = x(t_0)$$

why? $\delta(t-t_0)$ is $=0$ except at $t=t_0$

So $x(t)\,\delta(t-t_0) = 0$ except at $t=t_0$:

$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$.

Then plug in and use the δ-function property

This is the same "sifting property" from DT:

$$\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] = x[n]$$

So we can write $x(t)$ as an integral of scaled and shifted impulse functions:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

only $\neq 0$ for $\tau = t$ ...

As we did in the DT setting, let's define $h(t)$ to be the system response to $\delta(t)$.

Def

Def. The impulse response of a CT LTI system is the output of the system with input $x(t) = \delta(t)$.

Main

If

$$\delta(t) \longrightarrow h(t) \quad \text{impulse response}$$

then

$$\alpha\,\delta(t) \longrightarrow \alpha\,h(t)$$

What about delay? Use time invariance:

$$\alpha\,\delta(t-\tau) \longrightarrow \alpha\,h(t-\tau)$$

Set $\alpha = x(\tau)$ for example...

$$x(\tau)\,\delta(t-\tau) \longrightarrow x(\tau)\,h(t-\tau)$$

Note we are just using simple properties here...

Now, we saw that any function $x(t)$ can be written as an integral over δ-functions:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

So now we can "cheat" a bit (this is actually fine, mathematically, but differs from the linearity we saw earlier:

$$\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \longrightarrow \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

This means that the output $y(t)$ of a system with impulse response $h(t)$ to an input $x(t)$ is

$$\int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau.$$

Def

Theorem (convolution theorem for CT LTI Systems)

The output of a CT LTI system with impulse response $h(t)$ to an input signal $h(t)$ is

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau.$$

Not

The integral

$$\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

is called the convolution of $x(t)$ and $h(t)$ and is denoted by $(x * h)(t)$. The CT convolution is also symmetric:

$$(x * h)(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= (h * x)(t)$$

Try

Try to <u>prove</u> that CT convolution is symmetric. You can mimic the same argument from DT.

Phil

The book goes into a more formal/rigorous approach to the sifting property of δ-functions and analyzing an approximation of the CT convolution in terms of DT. This is <u>definitely</u> worth reading if/when you get confused about δ-functions and how they work. But for the first time around it's easier to just "integrate on both sides" and treat the integral like it is a sum. That is basically what the approximation argument shows.

Main

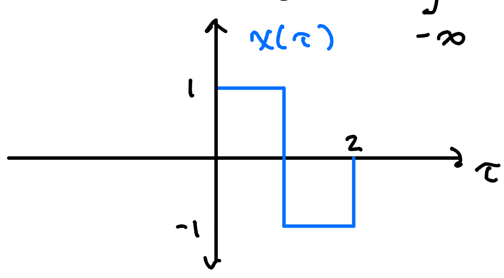
So for DT and CT LTI systems, the output signal is the convolution of the input signal with the impulse response.
Looking at the integral:

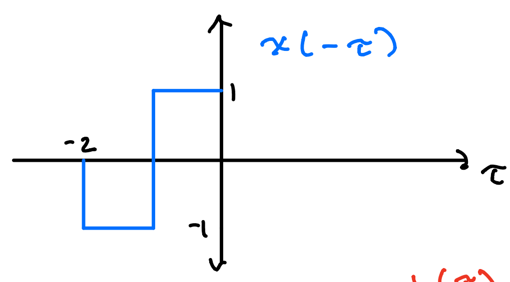
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

we can see the same interpretation: $y(t)$ is the integral (sum) of scaled and shifted impulse responses.

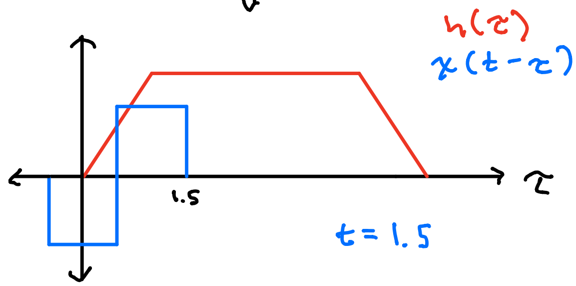
We can give a similar visual interpretation to the $h * x$ version...

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$





x(t) goes into the filter starting from time t=0, corresponding to a flip and then a shift forward as the signal goes in.



Ex

Example

$$h(t) = u(t)$$

$$x(t) = e^{-at}\, u(t) \qquad a > 0$$

Now we can compute with $h(\tau)\, x(t-\tau)$
or $x(\tau)\, h(t-\tau)$

Which is better? It depends on what kind of integral is less confusing. Let's do both!

But **first**: $x(t) = 0$ for $t \leq 0$ and
$h(t) = 0$ for $t \leq 0$.

This means:
$y(t) = 0$ for $t \leq 0$. (Why?)

So we just need to find $y(t)$ for $t > 0$:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-a(t-\tau)}\, u(\tau)\, u(t-\tau)\, d\tau$$

Plugging in the signals

$$= \int_{0}^{t} e^{-a(t-\tau)}\, d\tau$$

$u(\tau) = 0$ for $\tau \leq 0$
and
$u(t-\tau) = 0$ for $\tau \geq t$

$$= \left[\frac{1}{a} e^{a\tau} e^{-at} \right]_{\tau=0}^{t}$$

$$= \frac{1}{a} - \frac{1}{a} e^{-at}$$

$$= \frac{1}{a}\left(1 - e^{-at}\right)$$

Putting it together:

$$y(t) = \begin{cases} 0 & t \leq 0 \\ \frac{1}{a}(1-e^{-at}) & t \geq 0 \end{cases}$$

$$= \frac{1}{a}\left(1-e^{-at}\right) u(t)$$

What about the other way?

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-a\tau}\, u(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{0}^{t} e^{-a\tau}\, d\tau$$

same argument as above

$$= \left[-\frac{1}{a} e^{-a\tau} \right]_{\tau=0}^{t}$$

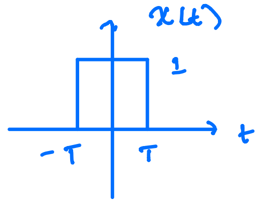
$$= \frac{1}{a}\left(1-e^{-at}\right)$$

Often one integral is a little easier/less confusing than the other. You have to build up your intuition to see which one will be better.

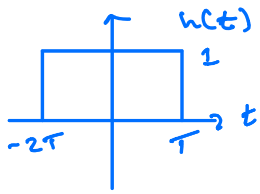
See Example 2.6 for some pictures.

Example: two boxcars:

$$x(t) = \begin{cases} 1 & |t| \le T \\ 0 & |t| > T \end{cases}$$



$$h(t) = \begin{cases} 1 & |t| \le 2T \\ 0 & |t| > 2T \end{cases}$$

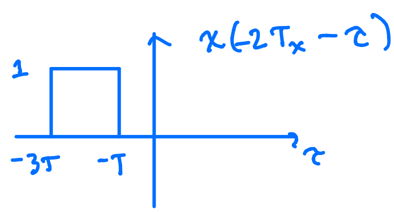


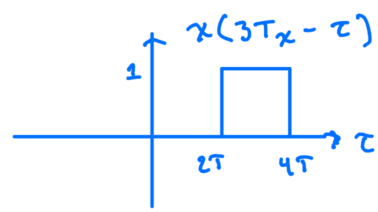
Let's try $h * x$ (you can try $x * h$ on your own).

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)$$

First: note $x(\tau) = x(-\tau)$.

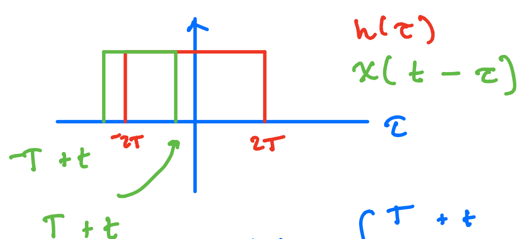
So $x(t-\tau)$ is $x(-(\tau - t))$:





1) for $t \le -3T$ $x(t-\tau)$ does not overlap $h(\tau)$ so $y(t) = 0$.

2) for $-3T < t \le -T$ there is a partial overlap:



$$y(t) = \int_{-2T}^{T+t} 1 \, d\tau = 3T + t$$

3) for $-T < t \le T$ there is total overlap:

$$y(t) = \int_{-T+t}^{T+t} 1 \, d\tau = 2T$$

4) for $T < t \leq 3T$ we get a similar overlap:

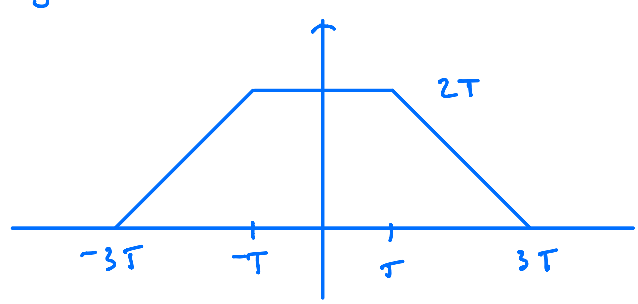
$$y(t) = \int_{-T+t}^{2T} 1 \, dt = 3T - t$$

5) for $t > 3T$ there is again no overlap so $y(t) = 0$.

Algebraically:

$$y(t) = \begin{cases} 0 & t \leq -3T \\ 3T + t & -3T < t \leq -T \\ 2T & -T < t < T \\ 3T - t & T \leq t < 3T \\ 0 & t \geq 3T \end{cases}$$

Graphically:



Why does this make sense? The system h acts like a "windowed integrator" — it chops off a 4T length chunk of the input signal and the convolution integrates that part.

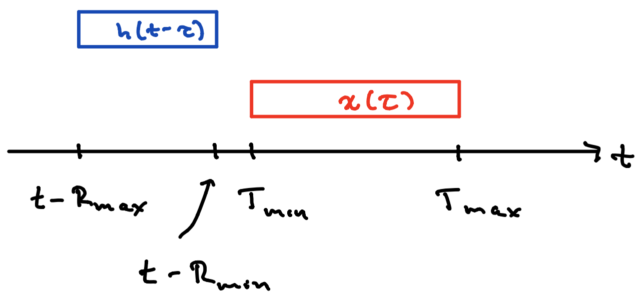
Main

Suppose 1) $x(t) = 0$ for $t < T_{min}$ and $t > T_{max}$

2) $h(t) = 0$ for $t < R_{min}$ and $t > R_{max}$

Then we can go through the same exercise that we did for DT signals:

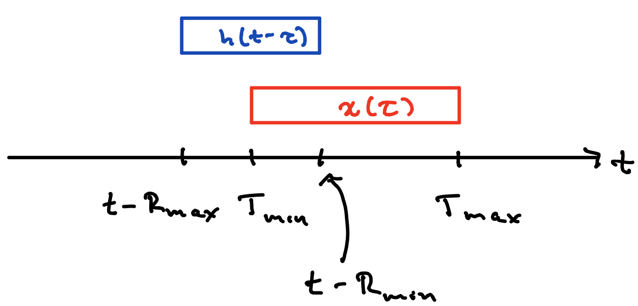
1)



So for $t - R_{min} \le T_{min}$ $\quad y(t) = 0$

or $t < R_{min} + T_{min} \Rightarrow y(t) = 0$

2)

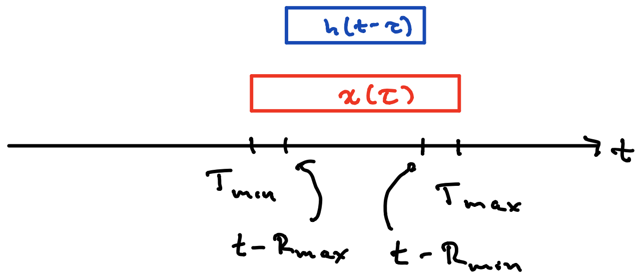


For $t - R_{max} < T_{min} \le t - R_{min}$

we have

$$y(t) = \int_{T_{min}}^{t - R_{min}} x(\tau)\, h(t-\tau)\, d\tau$$

3)



For $t - R_{max} > T_{min}$

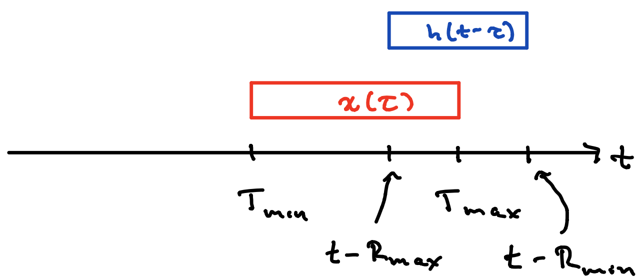
$t - R_{min} < T_{max}$

or $R_{max} + T_{min} < t < R_{min} + T_{max}$

we have

$$y(t) = \int_{t-R_{max}}^{t-R_{min}} x(\tau)\, h(t-\tau)$$

4)

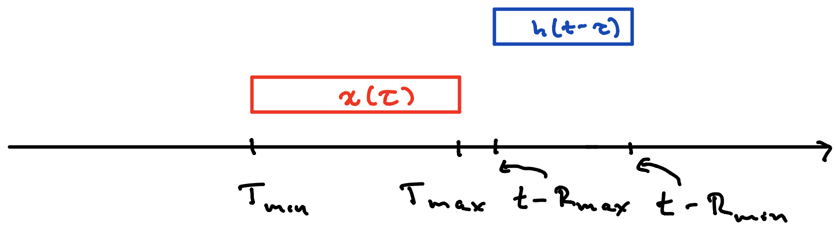


For $t - R_{max} \leq T_{max} < t - R_{min}$

or $R_{min} + T_{max} \leq t < R_{max} + T_{max}$

$$y(t) = \int_{t-R_{max}}^{T_{max}} x(\tau)\, h(t-\tau)$$

5)



For $\quad t - R_{max} \geq T_{max}$

or $\quad t \geq R_{max} + T_{max}$,

$$y(t) = 0$$

Ex

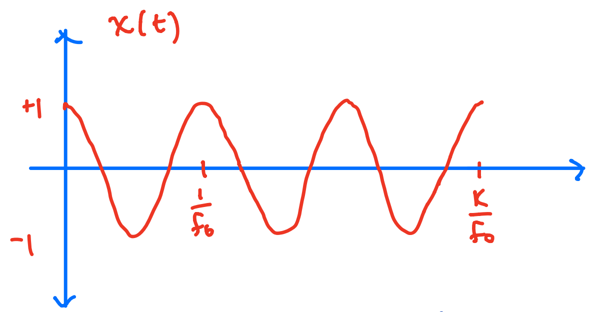
Example:

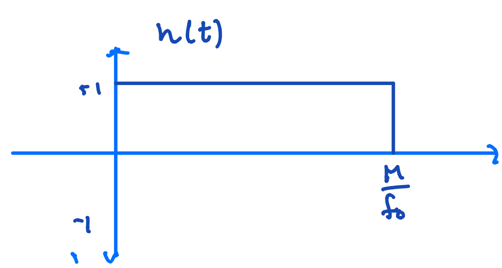
$$x(t) = \begin{cases} \cos(2\pi f_0 t) & 0 \leq t \leq \frac{K}{f_0} \\ 0 & \end{cases}$$

$$h(t) = u(t) - u(t - M/f_0)$$

Assume $M < K$

Remember: always draw a picture!!!





!!!

If this graph is at all confusing, stop and make sure you understand!!!

Now that we have a picture of the two functions we can do the integral:

$$y(t) = \int_{-\infty}^{\infty} \cos(2\pi f_0 \tau) \left(u(t-\tau) - u(t-\tau-M/f_0)\right) d\tau$$

$T_{min} = 0 \qquad T_{max} = K/f_0$

$R_{min} = 0 \qquad R_{max} = M/f_0$

We can use the general formula:

1) + 5) $y(t) = 0$ for $t \leq 0$ and $t \geq \frac{K+M}{f_0}$

2)

$$y(t) = \int_{T_{min}}^{t-R_{min}} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_0^t \cos(2\pi f_0 \tau)\, d\tau$$

$$= \left[\frac{1}{2\pi f_0} \sin(2\pi f_0 \tau)\right]_{\tau=0}^{t}$$

$$= \frac{1}{2\pi f_0} \sin(2\pi f_0 t)$$

3) for $R_{max} + T_{min} < t < R_{min} + T_{max}$

$$y(t) = \int_{t-R_{max}}^{t-R_{min}} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{t-M/f_0}^{t} \cos(2\pi f_0 \tau)\, d\tau$$

$$= \left[\frac{1}{2\pi f_0} \sin(2\pi f_0 \tau) \right]_{t-M/f_0}^{t}$$

$$= \frac{1}{2\pi f_0} \left(\sin(2\pi f_0 t) - \sin(2\pi f_0 (t - M/f_0)) \right)$$

integer multiple of 2π

$$= \frac{1}{2\pi f_0} \left(\sin(2\pi f_0 t) - \sin(2\pi f_0 t - \underline{2\pi M}) \right)$$

$$= \frac{1}{2\pi f_0} \left(\sin(2\pi f_0 t) - \sin(2\pi f_0 t) \right)$$

$$= 0$$

4) for $R_{min} + T_{max} \le t < R_{max} + T_{max}$

$$y(t) = \int_{t-R_{max}}^{T_{max}} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{t-M/f_0}^{K/f_0} \cos(2\pi f_0 \tau)\, d\tau$$

$$= \left[\frac{1}{2\pi f_0} \sin(2\pi f_0 \tau) \right]_{t-M/f_0}^{K/f_0}$$

$$= \frac{1}{2\pi f_0} \left(\sin\left(2\pi f_0 \frac{K}{f_0}\right) - \sin(2\pi f_0 (t - M/f_0)) \right)$$

$$\frac{1}{2\pi f_0} \left(\cancel{\sin(2\pi K)}^{\,0} - \sin(2\pi f_0 t - 2\pi M) \right)$$

$$= -\frac{1}{2\pi f_0} \sin(2\pi f_0 t)$$

Putting it all together:

$$y(t) = \begin{cases} 0 & t \leq 0 \\ \frac{1}{2\pi f_0} \sin(2\pi f_0 t) & 0 < t \leq \frac{M}{f_0} \\ 0 & \frac{M}{f_0} < t < \frac{K}{f_0} \\ \frac{-1}{2\pi f_0} \sin(2\pi f_0 t) & \frac{K}{f_0} \leq t < \frac{K+M}{f_0} \\ 0 & t \geq \frac{K+M}{f_0} \end{cases}$$

Try

As before, try getting the general form for the 5 regions to consider for the case where $h(t)$ is longer than $x(t)$.

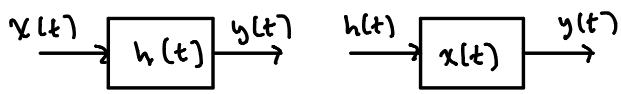
Phil

We don't use the terms FIR / IIR for CT systems but instead "time-limited" is used for the case where $h(t) \neq 0$ for $R_{min} < t < R_{max}$ only. Later on we will see that being time limited has implications for the frequency content of a signal or system. This turns out to be related to an <u>uncertainty principle</u> (like the famous Heisenberg Uncertainty Principle from physics).

Main

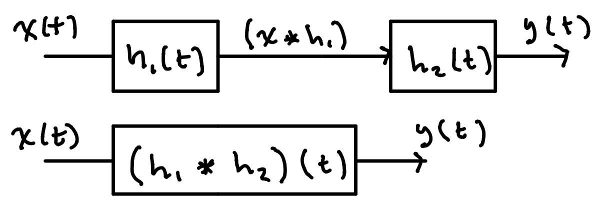
As with DT convolution, CT convolution is commutative, associative, and distributes over addition:

Commutative



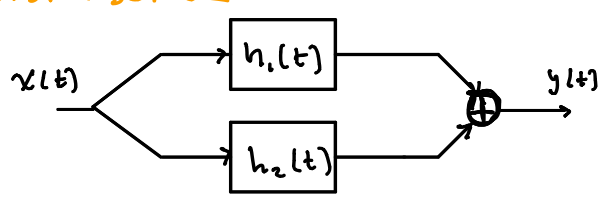
$$(x * h)(t) = (h * x)(t)$$

Associative

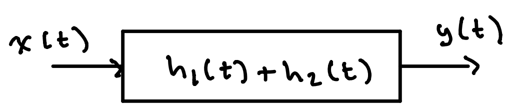


$$((x * h_1) * h_2)(t) = (x * (h_1 * h_2))(t)$$

Distributive



$$(x * h_1)(t) + (x * h_2)(t) = (x * (h_1 + h_2))(t)$$



These properties, plus time invariance, can simplify the analysis of CT systems just like they can for DT systems.

Ex

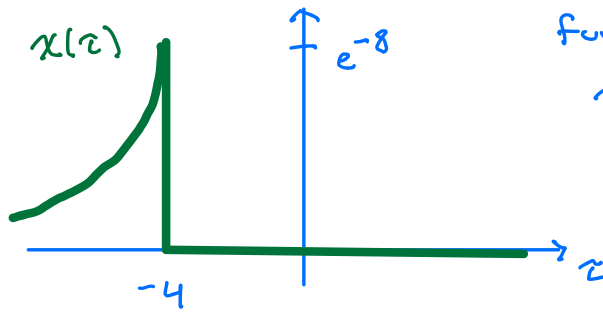
<u>Example</u> (modified from Example 2.8)

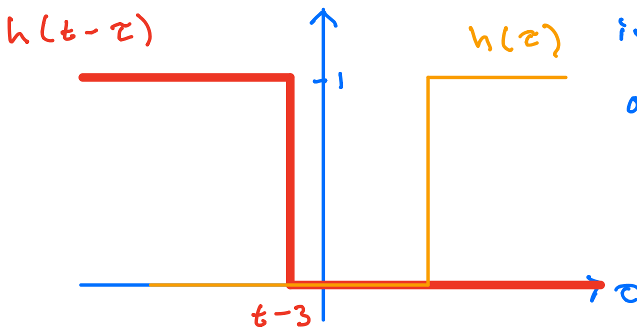
Let $x(t) = e^{-2t}\, u(-4-t)$

$h(t) = u(t-3)$

what is $y(t) = (x * h)(t)$?

First draw a picture with x and h as functions of τ — this is because in the convolution integral the integration is over τ





We can use time invariance to shift these back to 0:

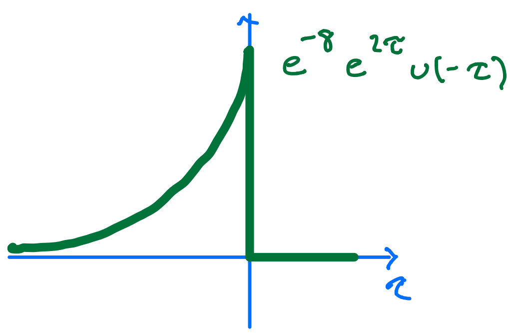
$$y(t) = x(t) * h(t)$$

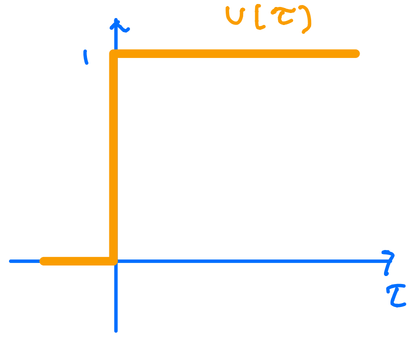
$$y(t+3) = x(t) * h(t+3) \quad \text{(time invariance)}$$

$$= h(t+3) * x(t) \quad \text{(commutativity)}$$

$$y(t-1) = h(t+3) * x(t-4) \quad \text{(time invariance)}$$

$$= u(t) * e^{2t} u(-t)$$





Define $z(t) = e^{-8} e^{2t} u(-t)$ $\qquad g(t) = u(t)$

So now do this convolution: $z * g$

$$(z*g)(t) = \int_{-\infty}^{\infty} e^{-8} e^{2\tau} u(-\tau)\, u(t-\tau)\, d\tau$$

$$= \begin{cases} \int_{-\infty}^{t} e^{-8} e^{2\tau}\, d\tau = \frac{1}{2} e^{-8} e^{2t} & \text{for } t < 0 \\ \int_{-\infty}^{0} e^{-8} e^{2\tau}\, d\tau = \frac{1}{2} e^{-8} & \text{for } t \geq 0 \end{cases}$$

Now shift back: this is $y(t-1)$ so we advance by 1 to get $y(t)$.

$$y(t) = \begin{cases} \frac{1}{2} e^{-8} e^{2(t+1)} & t < -1 \\ \frac{1}{2} e^{-8} & t \geq -1 \end{cases}$$

Main

The previous problem illustrates how to use commutativity and time invariance to shift the impulse response / signal to an easier-to-manipulate form.

- Commutative: $(x * h)(t) = (h * x)(t)$

Proof.

$$\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{\infty}^{-\infty} -x(t-s)\, h(s)\, ds$$

$$= \int_{-\infty}^{\infty} h(s)\, x(t-s)\, ds$$

Change of variables:

$s = t - \tau$
$\tau = t - s$
$d\tau = -ds$
τ goes from $-\infty$ to ∞
s goes from ∞ to $-\infty$

- Associative: $((x * h_1) * h_2)(t) = (x * (h_1 * h_2))(t)$

$$(x * h_1)(s) = \int_{-\infty}^{\infty} x(\tau)\, h_1(s-\tau)\, d\tau$$

$$((x * h_1) * h_2)(t)$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau)\, h_1(s-\tau)\, d\tau \right) h_2(t-s)\, ds$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)\, h_1(s-\tau)\, h_2(t-s)\, d\tau\, ds$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)\, h_1(s-\tau)\, h_2(t-s)\, ds\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} h_1(s-\tau)\, h_2(t-s)\, ds \right) d\tau$$

$r = s - \tau$, $t - s = (t-\tau) - r$

$dr = ds$

$$= \int_{-\infty}^{\infty} x(\tau)\, (h_1 * h_2)(t-\tau)\, d\tau$$

$$= (x * (h_1 * h_2))(t)$$

- Distributive:

$$(x * (h_1 + h_2))(t) = (x * h_1)(t) + (x * h_2)(t)$$

$$\int_{-\infty}^{\infty} x(\tau)\, (h_1(t-\tau) + h_2(t-\tau))\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h_1(t-\tau)\, d\tau + \int_{-\infty}^{\infty} x(t)\, h_2(t-\tau)\, d\tau$$

We can use these properties to simplify calculations by reducing problems to simpler subproblems.

Interpreting the impulse response

Much like the DT impulse response, we can interpret the impulse response in terms of what it <u>does</u>.

For example, $\delta(t - t_0)$ makes a copy of $x(t)$ starting at time t_0:

$$x(t) * \delta(t-t_0) = x(t-t_0)$$

This is also a <u>delay</u> by t_0

Extending this:

$$x(t) * (\delta(t-t_0) + 2\delta(t-t_1))$$

$$= x(t-t_0) + 2x(t-t_1)$$

= copy of x delayed by t_0
+
$2 \cdot$ copy of x delayed by t_1

Extending even more:

$$h(t) = \sum_{k=-\infty}^{\infty} \delta(t-t_k)$$

$$(x * h)(t) = \sum_{k=-\infty}^{\infty} x(t-t_k)$$

Doing some examples can help:

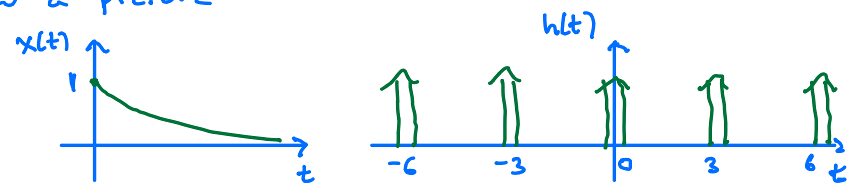
Ex

Example (Oppenheim & Willsky 2.12)

$$y(t) = \left(e^{-t} u(t)\right) * \sum_{k=-\infty}^{\infty} \delta(t-3k)$$

This system makes copies of $x(t)$ starting at integer multiples of 3

① Draw a picture



So the output is the sum of copies of $e^{-t}u(t)$ starting at integer multiples of 3.

The output at time t depends on past copies

② Use LTI system properties to simplify.

Maybe try distributive property:

$$y(t) = \sum_{k=-\infty}^{\infty} \left(e^{-t} u(t)\right) * \delta(t-3k)$$

delay by $3k$

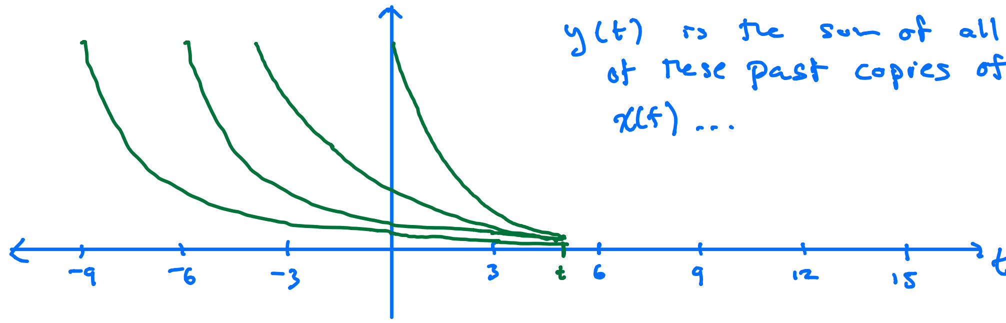
$$= \sum_{k=-\infty}^{\infty} e^{-(t-3k)} u(t-3k)$$

③ Do the calculation:

$$y(t) = \sum_{k=-\infty}^{\infty} e^{-(t-3k)} u(t-3k)$$

$= 0$ if $t < 3k$ or $k > 3/t$ so...

$$= \sum_{k=-\infty}^{\lfloor 3/t \rfloor} e^{-(t-3k)}$$



$y(t)$ is the sum of all of these past copies of $x(t)$ ...

$$y(t) = \sum_{k=-\infty}^{\lfloor 3/t \rfloor} e^{-(t-3k)}$$

$$= e^{-t} \sum_{k=-\infty}^{\lfloor 3/t \rfloor} e^{3k}$$

$$= e^{-t} \sum_{k=\lfloor 3/t \rfloor}^{\infty} e^{-3k}$$

$$= e^{-t} \frac{e^{\lfloor 3/t \rfloor}}{1 - e^{-3}}$$

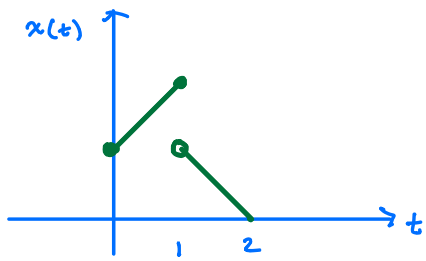
The problem in the book asks to show $y(t)$ is of the form Ae^{-t} for $0 \le t < 3$. In that range of t, $e^{\lfloor 3/t \rfloor} = e^0 = 1$, so

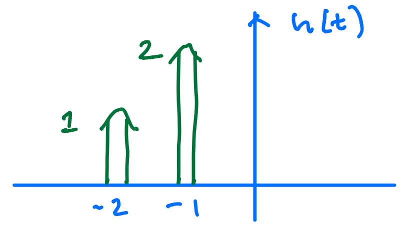
$$y(t) = \underbrace{\frac{1}{1-e^{-3}}}_{=A} e^{-t}$$

<u>Example</u>: Suppose
$$x(t) = \begin{cases} t+1 & 0 \le t \le 1 \\ 2-t & 1 < t \le 2 \\ 0 & \text{elsewhere} \end{cases}$$
and $h(t) = \delta(t+2) + 2\delta(t+1)$

To find $y(t) = x(t) * h(t)$ ...

① Draw a picture:





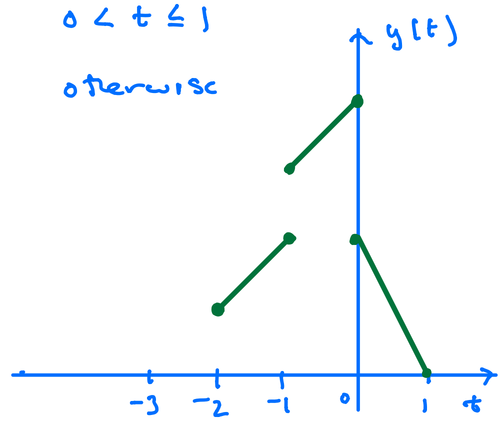
② Use convolution properties:

$$(x * h)(t) = x(t) * \delta(t+2)$$
$$+ 2x(t) * \delta(t+1)$$

$$x(t) * \delta(t+2) = x(t+2) = \begin{cases} t+3 & -2 \leq t \leq -1 \\ -t & -1 < t \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

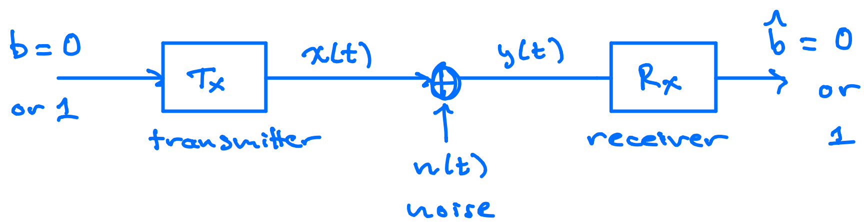
$$x(t) * 2\delta(t+1) = 2x(t+1) = \begin{cases} 2t+4 & -1 \leq t \leq 0 \\ 2-2t & 0 < t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$y(t) = \begin{cases} t+3 & -2 \leq t \leq -1 \\ -t+2t+4 & -1 < t \leq 0 \\ 2-2t & 0 < t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



A more substantive example:

In communication system, bits of information (0's and 1's) are encoded into CT waveforms for transmission. A simplified model looks like this:



Typically we think of a bit as being encoded into a "pulse" of duration T. For example:

1. (antipodal signaling) $b=0 \Rightarrow x(t) = -A\cos(2\pi f_0 t)$
   $b=1 \Rightarrow x(t) = A\cos(2\pi f_0 t)$

2. (orthogonal signaling) $b=0 \Rightarrow x(t) = A\sin(2\pi f_0 t)$
   $b=1 \Rightarrow x(t) = A\cos(2\pi f_0 t)$

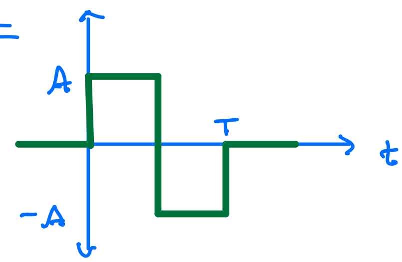
The receiver gets a noisy version of the signal $x(t)$:

$$y(t) = x(t) + n(t)$$

and has to determine if $b=0$ or $b=1$ was sent based on $y(t)$.
The way this is done is through what is called a matched filter.

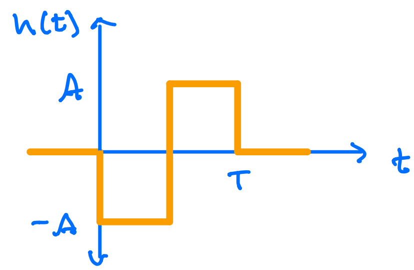
This is a filter "matched" to one of the pulses. The output $y(t)$ is filtered and then the receiver looks at the output at time T. For now let's focus on the case where there is no noise.

Suppose $x(t) =$

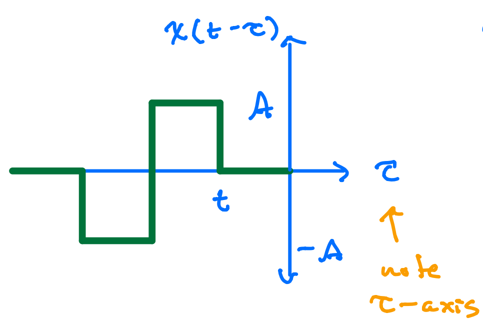


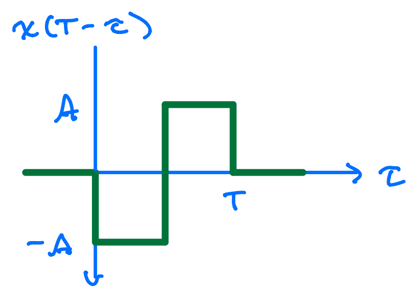
Then the matched filter is $h(t) = x(T-t)$:

Why do we call this matched?



If we look at $(x * h)(T)$, the flipped and slid version of $x(t)$ lines up perfectly with $h(t)$:

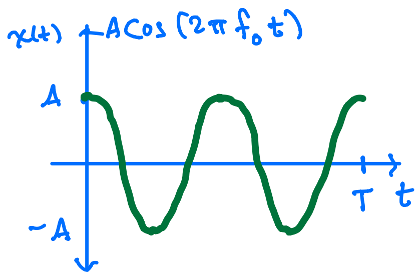


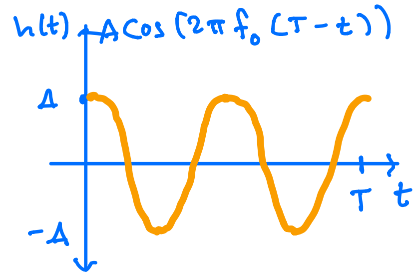


So what about our examples?

Suppose $T = K/f_0$ for some integer K.

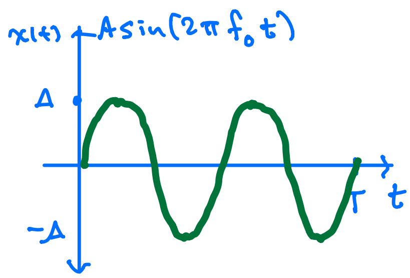
Then each sinusoid has an integer # of cycles since their periods are $1/f_0$

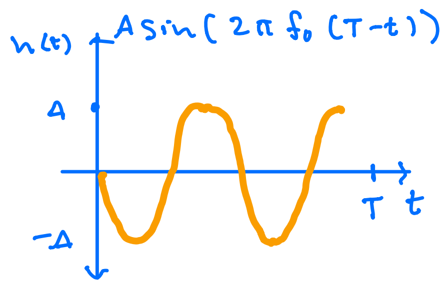




In algebra: $\cos(2\pi f_0 (T-t))$

$$
\begin{aligned}
&= \cos(2\pi f_0 T - 2\pi f_0 t) \\
&= \cos(2\pi K - 2\pi f_0 t) \\
&= \cos(-2\pi f_0 t) \\
&= \cos(2\pi f_0 t) \qquad \cos \text{ is } \underline{\text{even}}
\end{aligned}
$$

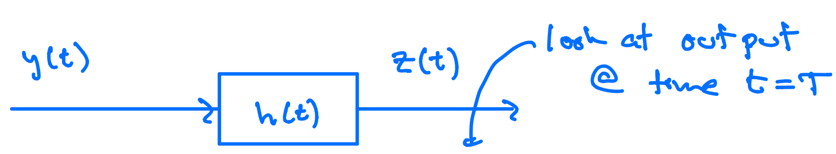




In algebra: $\sin(2\pi f_0 (T-t))$

$$
\begin{aligned}
&= \sin(2\pi f_0 T - 2\pi f_0 t) \\
&= \sin(2\pi K - 2\pi f_0 t) \\
&= \sin(-2\pi f_0 t) \\
&= -\sin(2\pi f_0 t) \qquad \sin \text{ is } \underline{\text{odd}}
\end{aligned}
$$

Let's focus on antipodal signaling. The receiver is doing this:



$$
h(t) = \begin{cases} A\cos(2\pi f_0 t) & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}
$$

Remember,

$$
y(t) = x(t) + n(t)
$$

So there are two components at the output of h:

$$z(t) = \underbrace{(x * h)(t)}_{\text{signal}} + \underbrace{(n * h)(t)}_{\text{noise}}$$

We are sampling the output at time $t = T$:

$$z(T) = \int_{-\infty}^{\infty} h(\tau)\, y(T-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, x(T-\tau)\, d\tau + \int_{-\infty}^{\infty} h(\tau)\, n(T-\tau)\, d\tau$$

$$= \int_0^T h(\tau)\, x(T-\tau)\, d\tau + \int_0^T h(\tau)\, n(T-\tau)\, d\tau$$

Focusing on the signal part, if $h(\tau)$ is the matched filter for $x(\tau)$ then $h(\tau) = x(T-\tau)$:

$$\int_0^T x(T-\tau)\, x(T-\tau)\, d\tau$$

$$= \int_0^T |x(T-\tau)|^2\, d\tau$$

$$= \int_0^T |x(\tau)|^2\, d\tau$$

$= \mathcal{E}_x$ the <u>energy</u> of $x(t)$.

For our antipodal signaling example, suppose we use a filter matched to $A\cos(2\pi f_0 t)$:

$$h(t) = A\cos(2\pi f_0 (T-t))$$
$$= A\cos(2\pi f_0 t)$$

If $b = 1$ then

$$x(t) = A\cos(2\pi f_0 t)$$

so the signal component at the receiver is

$$\int_{-\infty}^{\infty} x(\tau)\, h(T-\tau)\, d\tau$$
$$= \int_0^T A^2 \cos^2(2\pi f_0 \tau)\, d\tau$$

Not

We are going to use trig identities a lot in the rest of the class. Good things to remember:

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$= 2\cos^2\theta - 1$$
$$= 1 - 2\sin^2\theta$$

Is this a trigger warning?

$$\Rightarrow \cos^2\theta = \tfrac{1}{2}(1 + \cos(2\theta))$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

Sum and difference identities may also be useful.

So... $\int_0^T A^2 \cos^2(2\pi f_0 \tau)\, d\tau$

integer # of cycles

$$= \int_0^{K/f_0} \left(\tfrac{1}{2}A^2 + \tfrac{1}{2}A^2\cos(4\pi f_0 \tau)\right) d\tau$$

$$= \tfrac{1}{2}A^2 \frac{K}{f_0} = \tfrac{1}{2}A^2 T$$

If $b=0$ was sent then

$$\int_{-\infty}^{\infty} x(\tau)\, h(T-\tau)\, d\tau$$

$$= \int_0^T -A^2 \cos^2(2\pi f_0 \tau)\, d\tau$$

$$= -\frac{1}{2} A^2 T$$

So the signal part @ time $t=T$ at the output of the matched filter is

$$+\frac{1}{2}A^2T \quad \text{if} \quad b=1 \text{ was sent}$$

$$-\frac{1}{2}A^2T \quad \text{if} \quad b=0 \text{ was sent}$$

What about the noise? Well, that is a little beyond the scope of this class, but the end effect is that

$$z(T) = \begin{cases} \frac{1}{2}A^2T + \text{noise} & b=1 \\ -\frac{1}{2}A^2T + \text{noise} & b=0 \end{cases}$$

So to decide between $\hat{b}=1$ or $\hat{b}=0$ the receiver just checks the sign of $z(T)$:

$$z(T) \underset{\hat{b}=0}{\overset{\hat{b}=1}{\gtrless}} 0$$

This is a shorthand for

$$\hat{b} = \begin{cases} 1 & \text{if } z(T) > 0 \\ 0 & \text{if } z(T) < 0 \end{cases}$$

The energy in the noise determines the <u>signal-to-noise ratio</u> (SNR), which is an extremely important quantity for communication / DSP / machine learning applications

What about the orthogonal signaling? What if $b=0$ was sent so

$$x(t) = \begin{cases} A\sin(2\pi f_0 t) & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

If we use the $h(t)$ matched to $\cos(2\pi f_0 t)$:

$$\int_{-\infty}^{\infty} x(t)\, h(T-t)\, d\tau$$

$$= \int_0^T A^2 \sin(2\pi f_0 \tau)\cos(2\pi f_0 \tau)\, d\tau$$

integer # of cycles

$$= \int_0^{K/f_0} \frac{1}{2}\sin(4\pi f_0 \tau)\, d\tau$$

$$= \left[\frac{-1}{8\pi f_0} \cos(4\pi f_0 \tau) \right]_0^{K/f_0}$$

$$= 0$$

Phil + math

The reason this is called an orthogonal signaling scheme is because we can think of signals as vectors and a dot product between $x(t)$ and $y(t)$ is $\int_{-\infty}^{\infty} x(t)\, y(t)\, dt$

So in this way of thinking, $\sin(2\pi f_0 t)$ and

$\cos(2\pi f_0 t)$ have dot product $= 0$ which means they are at right angles to each other (i.e. orthogonal)

So for the orthogonal scheme:

$$z(T) = \begin{cases} \frac{1}{2}A^2 T + \text{noise} & b = 1 \\ 0 + \text{noise} & b = 0 \end{cases}$$

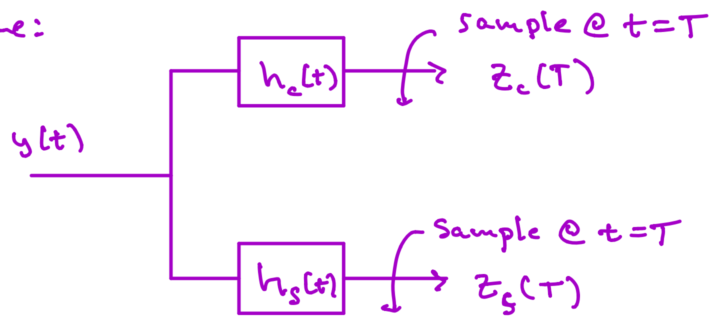
So it would seem like a good rule might be

$$\hat{b} = \begin{cases} 1 & z(T) > \frac{1}{4}A^2 T \\ 0 & z(T) < \frac{1}{4}A^2 T \end{cases}$$

but it turns out we can do better... to find out how, take ECE 322.

Try

Consider the following decoder for the orthogonal scheme:



We analyzed the upper branch. What is the lower branch $z_s(T)$ when $b = 0$ or 1 is sent?

Main

The previous example can give some insight into how signal & system <u>analysis</u> can help us understand engineering <u>design</u> by revealing tradeoffs. We have several parameters:

A — gain of the pulse

if $A \uparrow$ then $\mathcal{E}_x \uparrow$ so SNR $\uparrow$

→ more reliable decoding

→ higher power consumption

Large A also can be expensive in hardware — big heat sinks, nonlinearities, etc.

T — duration of the pulse

if $T \uparrow$ then $\mathcal{E}_x \uparrow$ → SNR $\uparrow$ (maybe)

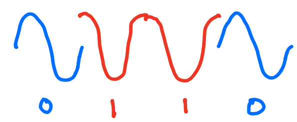
- <u>but</u> then we take more time to send 1 bit → lower rate.

$T = K/f_0$ — if $K \uparrow$ then more cycles, but for fixed T f_0 would increase:

→ oscillators can be <u>expensive</u>, not all f_0s are equally good

— we can lower f_0 and K but will still suffer from harmonics / other effects from switching.

That is, sending 0110 might look like



in hardware this abrupt switching can be hard due to issues with the speed of capacitors discharging, etc.

So what system analysis does is let us figure out what the key parameters are (here, A and T) in terms of performance (decoding errors, rate) to let us determine the cost-benefit tradeoffs (amplifiers, oscillators, analog-to digital converters, etc are expensive).