Discrete-Time Fourier Transforms

Objectives:

- Understand the DTFT and compute the transform for simple signals
- Use the DTFT properties to express the DTFT of more complex signals in terms of simpler ones
- Understand how periodic signals are represented in the DTFT
- Use the convolution property to find the output of an LTI system
- Use the multiplication property and periodic convolution to write the DTFT of products of DT signals
- Explain how aliasing arises in using the multiplication property.

Main

Filling out the picture

The story so far in the course is:

	Periodic	General	
Continuous	CTFS	CTFT	
Discrete	DTFS		

So we want to fill in the last box with a new transform (unsurprisingly) called the DTFT. This will let us understand how to represent DT signals in terms of their frequency components.

For the transforms we saw before,

Transform	Time Domain		Freq. domain
CTFT :	general signals $-\infty < t < \infty$	$\longleftrightarrow$	continuous function $-\infty < \omega < \infty$
CTFS :	periodic signals $0 \leq t < T$	$\longleftrightarrow$	discrete function $-\infty < k < \infty$
DTFS :	periodic signal $0 \leq n \leq N-1$	$\longleftrightarrow$	periodic function $0 \leq k \leq N-1$

The Discrete-time Fourier Transform (DTFT) maps a general DT time domain signal $x[n]$ into a <u>periodic</u> continuous function $X(e^{j\omega})$ with period 2π.

Not.

The notation $X(e^{j\omega})$ is used instead of $X(j\omega)$ or $X(\omega)$ to distinguish it as a DTFT. This is fairly confusing notation, however. It's best to think of it as $X(\omega)$.

Def

<u>Def</u>. The Discrete-Time Fourier Transform (DTFT) of a discrete time signal $x[n]$ is defined by the following analysis/synthesis pair:

"analysis" $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

"synthesis" $$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

!!!

The DTFT is different than the CTFT because in the frequency domain the DTFT is periodic with period 2π. To see this, look at the analysis equation:

$$X(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\omega+2\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}\, e^{-j2\pi n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$ since $e^{-j2\pi n} = 1$ for integer n

This shows the DTFT is periodic.

Phil

One way to think about <u>why</u> the DTFT is periodic is to think about DT signals as <u>sampled</u> verisons of

CT signals:

$$x[n] = x_c(nT_s)$$

where the sampling interval T_s tells us how often we are taking a sample. Measuring the frequency content of $x[n]$ is like measuring the frequency content of $x_c(t)$ indirectly — because the DT signal depends on the sampling frequency $f_s = 1/T_s$. So in a sense a frequency ω for $x[n]$ is measured relative to a frequency in $x_c(t)$.

Ex

$\boxed{\text{Example: } x[n] = a^n u[n] \quad |a| < 1}$

Then

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

$$= \frac{1}{1-ae^{-j\omega}} \qquad \text{Geometric series}$$

Going back:

$$x[k] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{1-ae^{-j\omega}} e^{j\omega k} d\omega$$

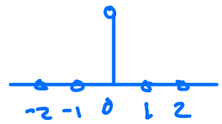
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=0}^{\infty} a^n e^{-j\omega(n-k)} d\omega$$

$$= \sum_{n=0}^{\infty} a^n \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega(n-k)} d\omega}_{=0 \text{ unless } n=k}$$

$$= a^k \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega(k-k)} d\omega$$

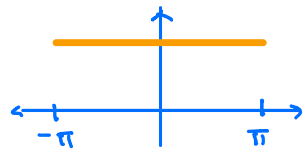
$$= a^k \quad \checkmark\checkmark$$

Example: $x[n] = \delta[n]$



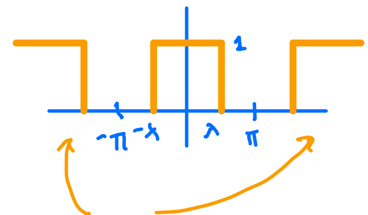
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = e^{-j\omega 0} = 1$$

We typically only graph one period of the DTFT



Example:

$$X(e^{j\omega}) = \begin{cases} 1 & 0 \le |\omega| < \lambda \\ 0 & \lambda \le |\omega| < \pi \end{cases}$$



remember: $X(e^{j\omega})$ is periodic with period 2π

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

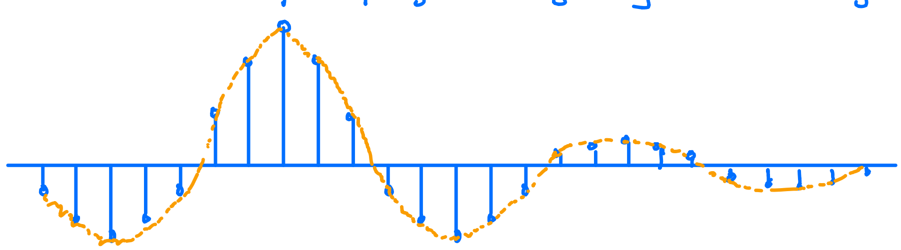
$$= \frac{1}{2\pi}\int_{-\lambda}^{\lambda} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{1}{jn} e^{j\omega n}\right]_{-\lambda}^{\lambda}$$

$$= \frac{1}{2\pi}\left(\frac{1}{jn} e^{j\lambda n} - \frac{1}{jn} e^{-j\lambda n}\right)$$

$$= \frac{1}{\pi n}\sin(\lambda n)$$

perhaps you are getting a sinc-ing feeling..



Main

With the DTFT we have to worry about convergence issues. That is,

$$\sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \longrightarrow \infty$$

It turns out there are at least two conditions which each guarantee convergence:

1) Absolute summability:

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

2) Square summability (= finite energy)

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

We can use these to show the DTFT does exist, not that it does not exist.

Ex

Example: $x[n] = u[n]$ is neither absolutely nor square summable. Does it have a DTFT???

Try

Show that $x[n] = \frac{1}{1+|n|}$ is square summable but not absolutely summable.

Main

What about the inverse transform?

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega t}\, dt$$

This integral is over a finite period of length 2π so we don't have any issues — compare this to the DTFS where only sum over 1 period.

DTFTs for periodic signals

Looking at the analysis equations for the DTFT and DTFS, they look similar:

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\omega_0 k n} \qquad \text{DTFS}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \qquad \text{DTFT}$$

If $x[n]$ is periodic with period N, what happens to $X(e^{j\omega})$?

Already we are in <u>trouble</u>: periodic signals have infinite energy! We had an issue like this with periodic signals for the CTFT — if $x(t)$ was periodic, $\mathcal{F}\{x(t)\}$ had some δ-functions. The same thing happens with the DTFS:

$$e^{j\omega_0 n} \xleftrightarrow{\mathcal{F}} \sum_{\ell=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0 - 2\pi \ell)$$

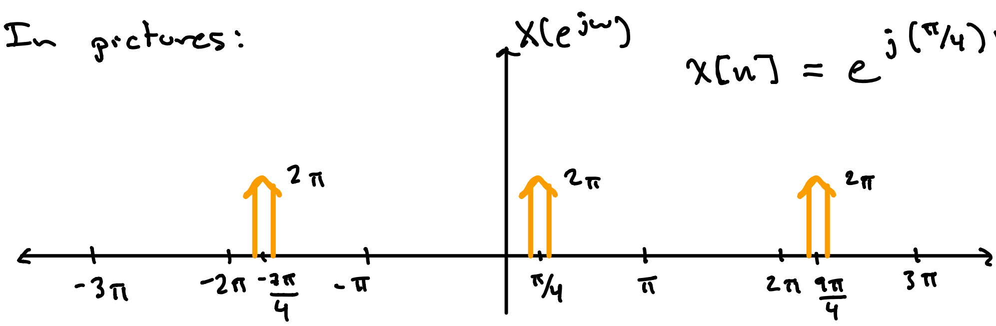
!!!

This is something special about the DTFT — complex exponentials transform into <u>impulse trains</u>. Since $X(e^{j\omega})$ is periodic, we can't have just one δ-function — they repeat every 2π.

Main

In pictures:

$x[n] = e^{j(\pi/4)n}$



To see why this is true, the inverse transform is easier since it involves only an integral over 2π:

$$x[n] = \int_{-\pi}^{\pi} \frac{1}{2\pi} \sum_{\ell=-\infty}^{\infty} 2\pi\,\delta(\omega-\omega_0-2\pi\ell)\, e^{j\omega n}\, d\omega$$

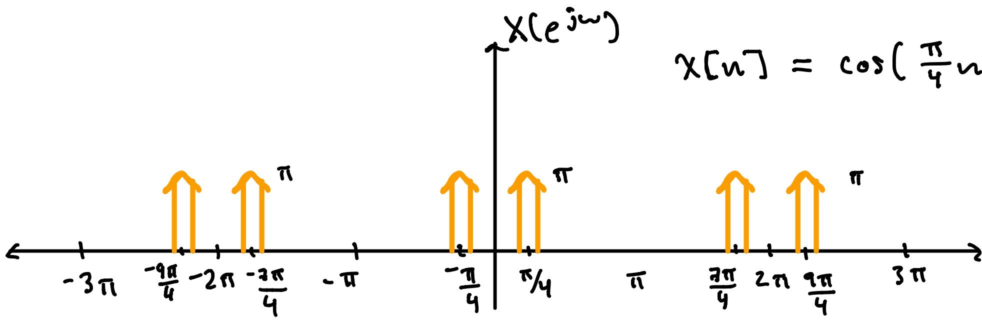
$$= \sum_{\ell=-\infty}^{\infty} \int_{-\pi}^{\pi} e^{j(\omega_0+2\pi\ell)n}\, \delta(\omega-\omega_0-2\pi\ell)\, d\omega$$

$$= \int_{-\pi}^{\pi} e^{j\omega_0 n}\, \delta(\omega-\omega_0)\, d\omega \quad \text{if } |\omega_0| < \pi$$

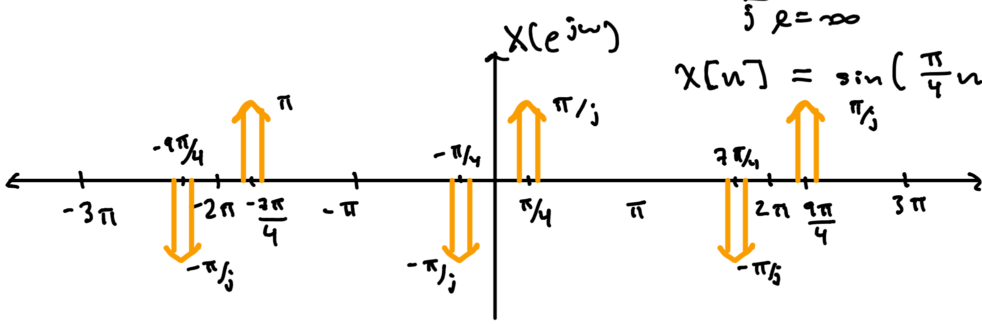
$$= e^{j\omega_0 n}$$

From this we immediately have sin and cos:

$$\cos(\omega_0 n) = \pi \sum_{\ell=-\infty}^{\infty} \delta(\omega-\omega_0-2\pi\ell) + \pi \sum_{\ell=-\infty}^{\infty} \delta(\omega+\omega_0-2\pi\ell)$$



$$\sin(\omega_0 n) = \frac{\pi}{j} \sum_{\ell=-\infty}^{\infty} \delta(\omega-\omega_0-2\pi\ell) - \frac{\pi}{j} \sum_{\ell=-\infty}^{\infty} \delta(\omega+\omega_0-2\pi\ell)$$



What if $x[n]$ is periodic with period N and has DTFS

$$a[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j\omega_0 k n} \quad ?$$

Go from the synthesis equation:

$$x[n] = \sum_{k=0}^{N-1} a[k] e^{j\omega_0 k n} \qquad \omega_0 = \frac{2\pi}{N}$$

$$X(e^{j\omega}) = \sum_{k=0}^{N-1} a[k] \sum_{\ell=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0 k - 2\pi \ell)$$

This has δ functions at every multiple of $\frac{2\pi}{N}$:

$$X(e^{j\omega}) = \sum_{m=-\infty}^{\infty} 2\pi\, a[m]\, \delta\left(\omega - \frac{2\pi m}{N}\right)$$

So from the DTFS we can get the DTFT.

Ex

Example: Suppose

$$x[n] = \sum_{m=-\infty}^{\infty} \delta[n - mN]$$

So this has impulses at multiples of N.

$$a[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk(2\pi/N)n}$$

$$= \frac{1}{N} \qquad \text{since } x[n] = 0 \text{ for } n = 1, 2, \ldots N-1$$

Thus $$X(e^{j\omega}) = \frac{2\pi}{N}\sum_{m=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi m}{N}\right)$$

Main

This last example shows an important property: the DTFT of a periodic impulse train in time is a periodic impulse train in frequency.

DTFT Properties

The DTFT has many of the same properties as the other transforms we have seen:

Linearity: $\mathcal{F}\{a\,x[n] + b\,y[n]\} = a\,X(e^{j\omega}) + b\,Y(e^{j\omega})$

Time and frequency shift:

$$\mathcal{F}\{x[n-n_0]\} = e^{-j\omega n_0} X(e^{j\omega})$$

$$\mathcal{F}\{e^{j\omega_0 n}\} = X(e^{j(\omega-\omega_0)})$$

Time reversal: $\mathcal{F}\{x[-n]\} = X(e^{-j\omega})$

If $x[n]$ is even, $x[n] = x[-n]$ so

$$X(e^{j\omega}) = X(e^{-j\omega}) \quad \text{is even}$$

If $x[n]$ is odd, $x[n] = -x[-n]$ so

$$X(e^{j\omega}) = -X(e^{-j\omega}) \quad \text{is odd}$$

Conjugation / Conjugate Symmetry:

$$\mathcal{F}\{x^*[n]\} = X^*(e^{-j\omega})$$

reversed + conjugation

So if $x[n]$ is real, $x[n] = x^*[n]$, so

$$X(e^{j\omega}) = X^*(e^{-j\omega})$$

Conjugate symmetric

If $x[n]$ is real and even

$$X(e^{j\omega}) = X^*(e^{j\omega}) \quad \text{is real and even}$$

If $x[n]$ is imaginary, $x[n] = -x^*[n]$, so

$$X(e^{j\omega}) = -X^*(e^{-j\omega})$$

conjugate anti symmetric

If $x[n]$ is real and odd,

$$X(e^{j\omega}) = -X^*(e^{j\omega})$$ is imaginary and odd

We have (for real $x[n]$):

$$x_{even}[n] \xleftrightarrow{\mathcal{F}} \mathrm{Re}\{X(e^{j\omega})\}$$
$$x_{odd}[n] \xleftrightarrow{\mathcal{F}} j\,\mathrm{Im}\{X(e^{j\omega})\}$$

Try

Try proving these properties yourself using the definitions and similar arguments as those in the CTFT / CTFS / DTFS.

Main

<u>First difference</u>: $\mathcal{F}\{x[n] - x[n-1]\}$?

This is easy using linearity and time shift:

$$\mathcal{F}\{x[n] - x[n-1]\} = (1 - e^{-j\omega})X(e^{j\omega})$$

Now what about an accumulator:

$$y[n] = \sum_{m=-\infty}^{n} x[m]$$

We have an issue since this looks not summable/stable as a system. We can get around this with a δ-function:

$$Y(e^{j\omega}) = \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{\ell=-\infty}^{\infty} \delta(\omega - 2\pi\ell)$$

DC value

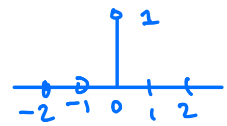
impulse train to make things periodic

Ex

Before we had some issue with $x[n] = u[n]$ — it was neither summable nor square summable. We can use our knowledge of systems to help.

$u[n]$ is the output of an accumulator with input $\delta[n]$:

$g[n] = \delta[n] =$



$$u[n] = \sum_{m=-\infty}^{n} \delta[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$

So

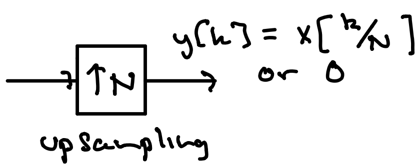
$$x[n] = \sum_{m=-\infty}^{n} g[n]$$

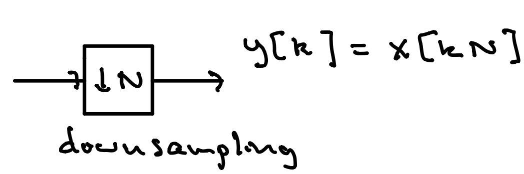
$$X(e^{j\omega}) = \frac{1}{1-e^{-j\omega}} G(e^{j\omega}) + \pi G(e^{j\cdot 0}) \sum_{\ell=-\infty}^{\infty} \delta(\omega - 2\pi\ell)$$

$$= \frac{1}{1-e^{-j\omega}} \, 1 + \pi \sum_{\ell=-\infty}^{\infty} \delta(\omega - 2\pi\ell)$$

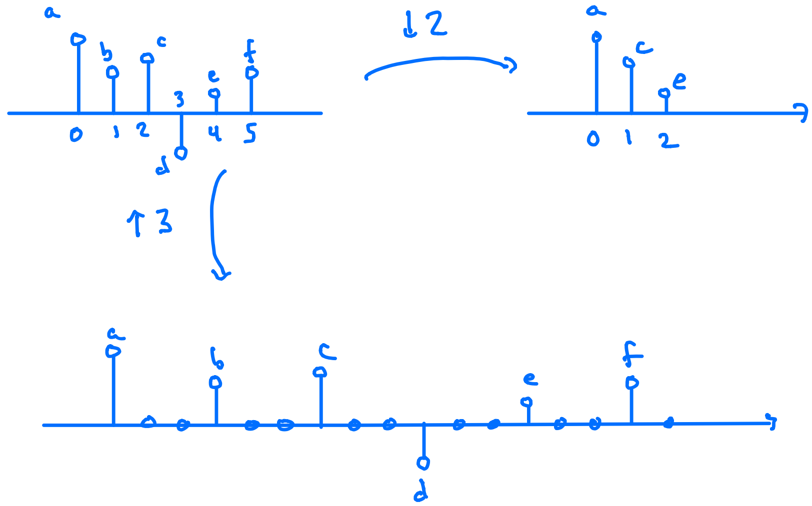
Upsampling and Downsampling

DT signals can't be squished and stretched in time like CT signals. Two core operations in DSP are upsampling and downsampling:





These are easy to visualize:



For upsampling

$$y[k] = \begin{cases} x[k/N] & \text{if } k \bmod N = 0 \\ 0 & \text{otherwise} \end{cases}$$

For downsampling

$$y[k] = x[kN]$$

Try: What happens when you downsample $a^n u[n]$ by a factor of 2? A factor of 3? A general factor of l?

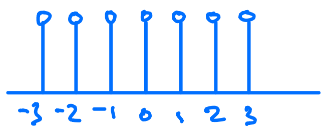
Downsampling <u>destroys</u> information in the signal, so in general we cannot say as much about the DTFT of a downsampled signal (yet). For upsampling, however:

$$y[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega N})$$

That is, if $x[n]$ is upsampled by a factor of N then the spectrum is squished by a factor of N.

Ex

Example: $x[n] = u[n+3] - u[n-4]$



a box between -3 and 3

We might suspect this FT's into a sinc... but a sinc is not periodic!

$$X(e^{j\omega}) = \sum_{n=-3}^{3} x[n]\, e^{-j\omega n}$$

$$= 1 + 2\cos(\omega) + 2\cos(2\omega) + 2\cos(3\omega)$$

seems good but if it was a box from $-N$ to N this might get ugly...

$$= \sum_{n=-3}^{3} e^{-j\omega n}$$

$$= e^{j\omega 3} \sum_{n=0}^{6} e^{-j\omega n}$$

trying to get a geometric series

$$= e^{j\omega 3} \sum_{n=0}^{6} \left(e^{-j\omega}\right)^n$$

$$= e^{j\omega 3}\, \frac{1 - e^{-j7\omega}}{1 - e^{-j\omega}}\; \frac{e^{j\omega/2}}{e^{j\omega/2}}$$

another trick...

$$= \frac{e^{j\omega(3+1/2)} - e^{-j\omega(3+1/2)}}{e^{j\omega/2} - e^{-j\omega/2}}$$

$$= \frac{\sin\left(\tfrac{7}{2}\omega\right)}{\sin\left(\tfrac{1}{2}\omega\right)}$$

Main

This is the discrete analogue of the sinc function. Since the DTFT is periodic, we need this function to be periodic. See the text (Ex 5.3) for a picture. In general

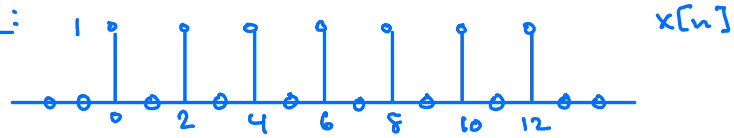
$$x[n] = \begin{cases} 1 & |n| \leq N \\ 0 & \text{otherwise} \end{cases} \quad \xleftrightarrow{\mathcal{F}} \quad X(e^{j\omega}) = \frac{\sin\left(\left(N+\tfrac{1}{2}\right)\omega\right)}{\sin\left(\tfrac{1}{2}\omega\right)}$$

Try

Prove this general formula using the tricks used in the example for $N=3$.

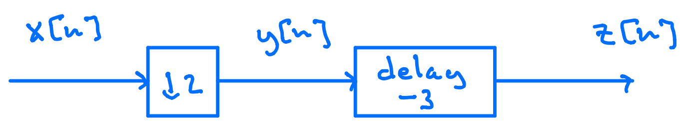
Ex

<u>Example:</u>



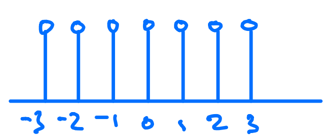
Find $X(e^{j\omega})$.

We can manipulate $x[n]$ into a form that we recognize:

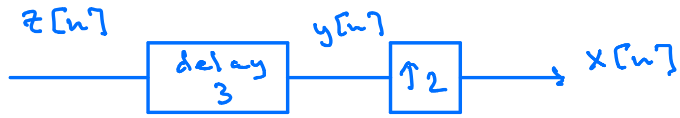


Then

$z[n] =$

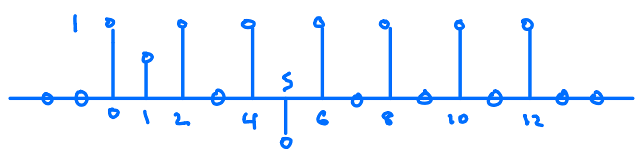


We need to find $Z(e^{j\omega})$ and reverse these operations:



!!! The thing to be careful about is if these steps are <u>reversible</u>. In particular, we need the reverse system to recover our original $x[n]$

This works in this case, but if we had $x[n]$ as this signal:



Then the downsampler would make it such that $z[n]$ would lose the values at 1 and 5, so the reverse system would not recover $x[n]$!

We already know

$$Z(e^{j\omega}) = \frac{\sin(7/2\,\omega)}{\sin(\omega/2)}$$

so

$$y[n] = z[n-3] \longleftrightarrow Y(e^{j\omega}) = e^{-j3\omega}\frac{\sin(\frac{7}{2}\omega)}{\sin(\frac{1}{2}\omega)}$$

$$x[n] = \text{upsampled } y[n] \text{ by } 2 \longleftrightarrow X(e^{j\omega}) = e^{-j6\omega}\frac{\sin(7\omega)}{\sin(\omega)}$$

Phil

Upsampling slows a signal down and so makes all the frequencies lower by a multiplicative factor. This is different than a frequency shift, which moves frequencies by an additive factor.

Main

Differentiation in frequency

If we look at $\frac{d}{d\omega} X(e^{j\omega})$, we get

$$\frac{d}{d\omega} X(e^{j\omega}) = \frac{d}{d\omega} \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] \frac{d}{d\omega} e^{-j\omega n}$$

$$= -j \sum_{n=-\infty}^{\infty} x[n]\, n\, e^{-j\omega n}$$

$$= -j \mathcal{F}\{ n\, x[n] \}$$

Thus

$$\mathcal{F}\{ n\, x[n] \} \longleftrightarrow j \frac{d}{d\omega} X(e^{j\omega})$$

Parseval's Relation

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega$$

total energy in $x[n]$ (time domain)

average power in $X(e^{j\omega})$ (frequency domain)

To see a nice example of how to use these properties, see Example 5.10 in the book.

Convolution and Multiplication

And now for our most important fact in the course:

Convolution in time	$\xleftrightarrow{\mathcal{F}}$	Multiplication in frequency
$y[n] = (x * h)[n]$	$\xleftrightarrow{\mathcal{F}}$	$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$

Multiplication in time	$\xleftrightarrow{\mathcal{F}}$	Periodic convolution in frequency
$y[n] = x[n]\,h[n]$	$\xleftrightarrow{\mathcal{F}}$	$Y(e^{j\omega}) = \frac{1}{2\pi}(X \circledast H)(e^{j\omega})$
		really: $Y(\omega) = \frac{1}{2\pi}(X \circledast H)(\omega)$

Ex

Example: Suppose we have an LTI filter:

$$h[n] = \frac{1}{\pi n} \sin\left(\frac{\pi}{4} n\right).$$

What is the output of this filter to the following inputs?

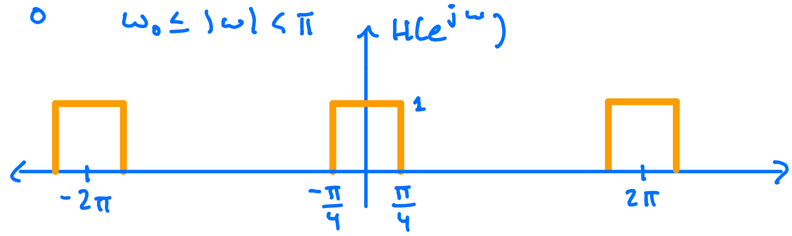
a) $x[n] = a^n u[n] \qquad |a| < 1, \; a$ real

b) $x[n] = \sin\left(\frac{\pi}{2} n\right) + \cos\left(\frac{\pi}{8} n\right)$

c) $x[n] = \dfrac{\sin\left(\pi/6 \; n\right)}{n}$

We have the following DTFT for $h[n]$

$$H(e^{j\omega}) = \begin{cases} 1 & 0 \le |\omega| < \omega_0 \\ 0 & \omega_0 \le |\omega| < \pi \end{cases}$$



So this is a lowpass filter with cutoff $\pi/4$.

Since the system output is $(x * h)[n]$, we just need to calculate the product of the transforms:

a) $$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$$Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega}) = \begin{cases} \dfrac{1}{1-ae^{-j\omega}} & |\omega| < \pi/4 \\ 0 & \text{otherwise} \end{cases}$$

$$y[n] = \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} \frac{1}{1-ae^{-j\omega}}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \sum_{\ell=0}^{\infty} \int_{-\pi/4}^{\pi/4} (ae^{-j\omega})^{\ell}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \sum_{\ell=0}^{\infty} \int_{-\pi/4}^{\pi/4} a^{\ell}\, e^{j\omega(n-\ell)}\, d\omega$$

$$= \frac{1}{2\pi} \sum_{\substack{\ell=0 \\ \ell \ne n}}^{\infty} a^{\ell} \left[\frac{1}{j(n-\ell)}\, e^{-j\omega(n-\ell)} \right]_{-\pi/4}^{\pi/4}$$

$$+ \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} a^{n}\; 1\; d\omega$$

$$= \frac{1}{4} a^n + \sum_{\substack{\ell=0 \\ \ell \neq n}}^{\infty} a^\ell \frac{1}{j(n-\ell)} \left(e^{-j\pi/4\,(n-\ell)} - e^{j\pi/4\,(n-\ell)} \right)$$

$$= \frac{1}{4} a^n + \sum_{\substack{\ell \neq n \\ \ell=0}}^{\infty} \frac{-1}{n-\ell}\, 2\, a^\ell \sin\left(\frac{\pi}{4}(n-\ell)\right)$$

b) $$X(e^{j\omega}) = \pi \sum_{\ell=-\infty}^{\infty} \delta\left(\omega - \frac{\pi}{2} - 2\pi\ell\right) + \pi \sum_{\ell=-\infty}^{\infty} \delta\left(\omega - \frac{\pi}{8} - 2\pi\ell\right)$$

The first sum are δ functions at

$\ldots -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$

$\longrightarrow$ cutoff is at $\pm\frac{\pi}{4}$ so none survive

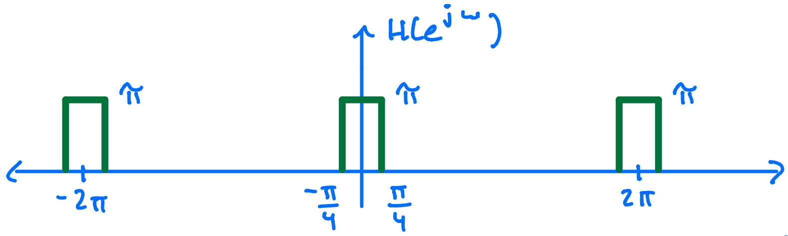
The second sum are δ-functions at

$\ldots, -\frac{7\pi}{8}, \frac{\pi}{8}, \frac{9\pi}{8}, \ldots$

$\longrightarrow$ $\pi/8$ is inside the cutoff $\rightarrow$ all survive

$$Y[n] = \cos\left(\frac{\pi}{8} n\right)$$

c) $X(e^{j\omega})$ is a lowpass filter with height π and cutoff $\pi/6$:



So the filter survives: $Y(e^{j\omega}) = X(e^{j\omega})$

and

$$y[n] = \frac{\sin(\pi/6\, n)}{n}$$

Main

For multiplication, we have that

$$x[n]\, h[n] \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} X(e^{j\omega}) \circledast H(e^{j\omega})$$

but so far we have only defined periodic convolution for discrete signals. But for continuous functions it works similarly:

$$X(e^{j\omega}) \circledast H(e^{j\omega}) = \int_{\langle 2\pi \rangle} X(e^{j\varphi})\, H(e^{j(\omega-\varphi)})\, d\varphi$$

We can take the integral over any interval of length 2π since both $X(e^{j\omega})$ and $H(e^{j\omega})$ are periodic with period 2π

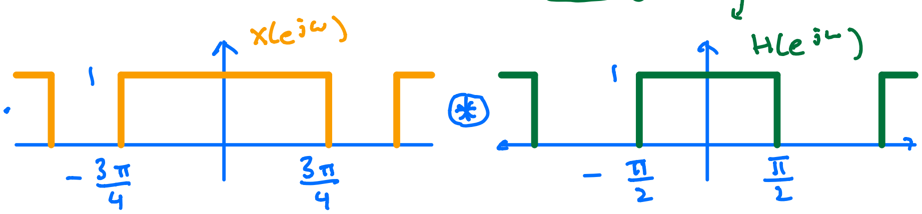
This looks like a regular convolution except we restrict the integral to be over any interval of length 2π. This convolution is not so hard — just more integrals.

Ex

<u>Example</u>: Let's take the product of two LPFs:

$$y[n] = \underbrace{\frac{\sin\left(\frac{3\pi}{4} n\right)}{\pi n}}_{x[n]} \cdot \underbrace{\frac{\sin\left(\frac{\pi}{2} n\right)}{\pi n}}_{= h[n]}$$

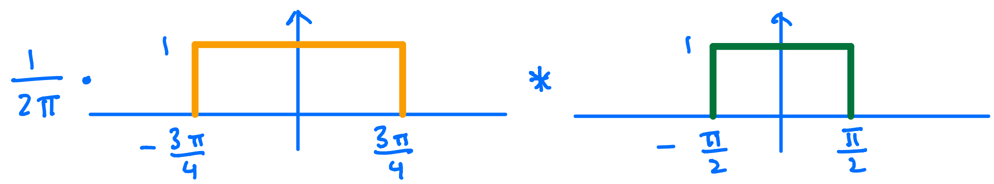
$$Y(e^{j\omega}) = \frac{1}{2\pi} \cdot$$

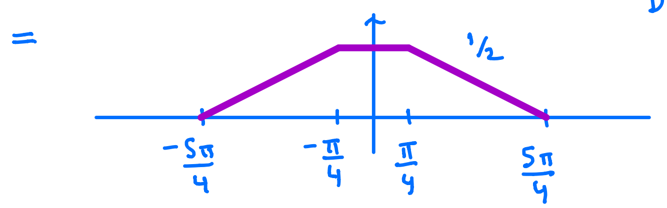


The way to do this is to convolve one period of each and then make copies @ period 2π

and then add them back up.

① Convolve one copy of each: (don't forget the $\frac{1}{2\pi}$)

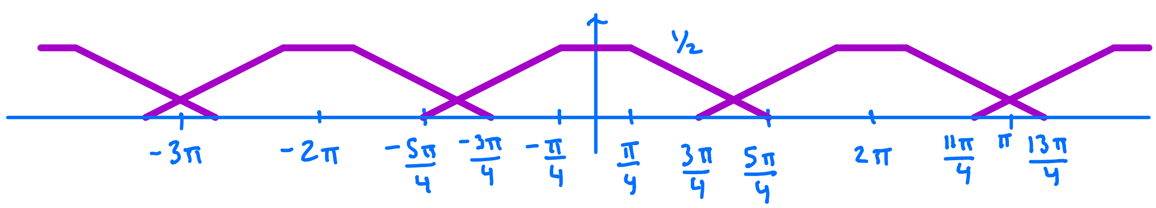




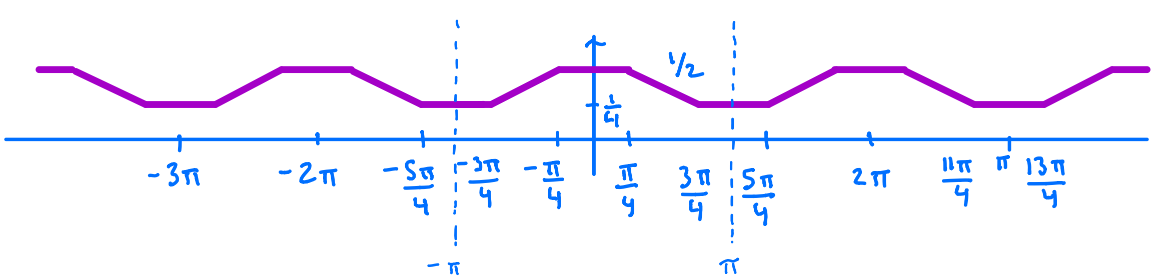
Note this extends beyond $[-\pi, \pi]$

Doing this graphically — this is why being comfortable with convolution is important!

② Make copies every 2π:



Note that because convolving one copy gives something that extends beyond $[-\pi, \pi]$, the periodic repetition <u>overlaps</u>.



This overlap between copies is called <u>aliasing</u>.

It's a major issue in DSP.

Try

Try a couple of periodic convolutions for yourself. Start with LPFs like those above. Suppose

$x_1[n]$ = LPF with cutoff α

$x_2[n]$ = LPF with cutoff β

Find a condition on α, β such that there is <u>no</u> aliasing in $y[n] = x_1[n]\, x_2[n]$.

Main

Duality & relationship to other transforms

As with the CTFT, we have some similarities in going from $t \to \omega$ and $\omega \to t$:

$$x[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$$

$$e^{j\omega_0 n} x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)})$$

Similarly, if we know a transform pair, sometimes an analogous pair holds in reverse.

DTFT and DTFS

If $x[n]$ is periodic with period N, then it has a DTFS $a[k]$ for $\omega_0 = \frac{2\pi}{N}$:

$$a[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\omega_0 n}$$

But now $a[k]$ is a periodic discrete sequence so it has a DTFT:

$$A(e^{j\omega}) = \sum_{k=-\infty}^{\infty} a[k]\, e^{-j\omega n}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} \sum_{l=-\infty}^{\infty} 2\pi\; x[n]\; \delta(\omega - n\omega_0 - 2\pi l)$$

This puts a train of δ-functions at frequencies $\frac{2\pi n}{N}$ for all n, with $x[n]\cdot\frac{2\pi}{N}$

So the DTFT of the DTFS of $x[n]$ makes a copy of the DT periodic signal $x[n]$ into the 2π periodic signal $A(e^{j\omega})$.

<u>DTFT and CTFS</u>: The DTFT looks a bit like a reverse version of a CTFS:

<u>CTFS</u>	<u>DTFT</u>
$x(t) = \sum_{k=-\infty}^{\infty} a[k]\, e^{jk\omega_0 n}$	$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$
$a[k] = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$	$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$

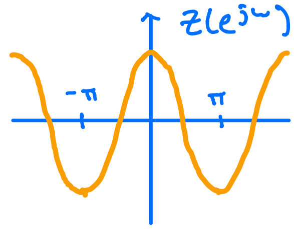
What if we look at $y[n] = x[-n]$? Then

$$y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{-j\omega n}\, d\omega$$

which is the CTFS of $X(e^{j\omega})$!

Ex

<u>Example:</u> Suppose $Z(e^{j\omega}) = \cos(\omega)$

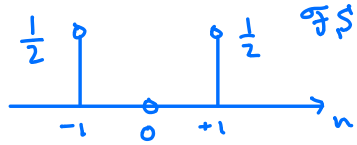


What is $z[n]$?

We know that CTFS of $Z(r) = \cos(r)$ is $z[-n]$

So: the CTFS of $\cos(r) = \frac{1}{2} e^{jr} + \frac{1}{2} e^{-jr}$

is



$\mathcal{FS}\{\cos(r)\}$

by direct inspection.

Letting $y[n] = \frac{1}{2}\delta[n+1] + \frac{1}{2}\delta[n-1]$

be this signal, we see that

$$z[-n] = y[n]$$

or

$$z[n] = y[-n] = \frac{1}{2}\delta[n+1] + \frac{1}{2}\delta[n-1]$$

Not much changed.

How about $Z(e^{j\omega}) = \sin(\omega)$?

Then $y[n] = \frac{-1}{2j}\delta[n+1] + \frac{1}{2j}\delta[n-1]$

$$z[n] = \frac{1}{2j}\delta[n+1] - \frac{1}{2j}\delta[n-1].$$